MTA Számitástechnikai Központja   Computing Centre of the Hungarian Academy of Sciences

TWO-LEVEL PLANNING

by

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INTRODUCTION[1]

In recent years work has begun in Hungary on the application of mathematical methods to the higher levels of planning. Experiments are proceeding in two directions. One of these has been the use of mathematical programming in several sectors of industry, to form a basis for their plans. The calculations - some of which have been concluded, while others are near completion - use economic optimum criteria to determine the most favourable program for the economic activities /production, producers' utilization, exports, imports, investments, etc./, of the whole of the particular sector of industry.[2]

The other direction has been the use of input-output tables /static Leontief models/ in national planning.[3] The National Planning Bureau now makes regular use of the input-output matrix of the economy to check the inner coordination of the annual and five-year plans. This is the first mathematical tool to have been used in Hungary for the preparation of macroeconomic plans. It is, however, as is generally known, not suitable for optimization and

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[1] The authors first published the method treated in this paper in duplicated form under the aegis of the Computing Centre of the Hungarian Academy of Sciences in May, 1962 [13] , and with the addition of a revised version of the mathematical part, in October, 1962 [17] .
An earlier version of this paper in Hungarian is under press in the Publications of the Mathematical Institute of the Hungarian Academy of Sciences [14] . A paper by Lipták [18] discusses a further developed version of the "general model" treated here.

[2] See [12] .

[3] Detailed information on the use of input-output tables in Hungary is presented in the material of the scientific conference held in Budapest in 1961, see [6] .

thus only destined to achieve the correct proportions between sectors.

A survey of the situation thus logically leads to the next step - the need to elaborate procedures that will permit optimization, but this time for the whole of the national economy. This is a requirement frequently voiced by the practical planners, and the Hungarian literature of the subject also contains such proposals. The ideas advanced so far have, however, not been able to overcome the basic difficulty of solving this task - that of either setting up a highly aggregated programming model in which case the freedom of choice is extremely narrow and the extent of aggregation, the excessive simplifications, endanger the utility of the computed results, or else setting up a very large sized model which will be free of these deficiencies, but in this case not even high power electronic computers will be able to cope with the numerical solution of the problem.

The present research project has been aimed at surmounting this very difficulty. The solution must obviously be sought in decomposing the large programming scheme. This idea has appeared several times in the literature of macroeconomic planning - it is sufficient to refer to the work of Kantorowich [10] , of Frisch [5] , and to the paper of Trzeciakowski [24] . Mathematical methods are, moreover, known for the decomposition of linear programming tasks of special kinds, e.g. in the papers of Dantzig and Wolfe [3] , [4] . Nevertheless it has been found that the known procedures do not provide solutions to the problem. Thus if the decomposition procedure of Dantzig and Wolfe were applied to the concrete macroeconomic model in hand, the "coordinating program" which figures there would still be of such size as to be unmanagable for computing with the usual processes /e.g. the simplex method/. Another approach to the solution was therefore adopted.

The planning task may originally be formulated as a single linear programming problem of the maximizing type, whose size will be too great for the given computing facilities. This we shall henceforth call the overall central information problem /OCI problem for short/. The OCI problem may be decomposed into sub-problems that can be solved by mutually independent "sectors", coordinated by the "centre" through having the later allocate the limitations prescribed in the OCI problem /the resources, materials and labour/ to the various sectors. The original OCI problem is then transformed into a two-level problem, in which the "central problem" is to evolve an allocation pattern where the sum of the maximal yields of the "sector problems" will be greatest. The solution of the two-level problem is achieved by setting up a game-theoretical model. The players are on the one hand the centre, on the other the team of sectors. The strategies of the centre are the feasible allocation patterns, those of the sectors are the feasible shadow price systems in the duals of the sector problems. The pay-off function is the sum of the dual sector objective functions. It is shown that if certain regularity conditions are satisfied, then the value of the polyhedral game which has thus been defined is the maximal yield of the OCI problem, and that with the help of its optimal strategies the solution of the OCI problem may be obtained. In place of a direct solution of the polyhedral game, a fictitious play of the game is undertaken. In the course of this each sector separately evaluates the suitably chosen initial allocations of the centre /by means of dual linear programming/, and reports back to the centre. The centre, applying a certain procedure, correspondingly modifies its initial allocations and sends down new directives to the sectors. The latter again evaluate these, report back to the centre; and so on. The iteration thus obtained permits the OCI problem to be solved with any required degree of accuracy, in the sense that a sufficient number of iterations will lead to a feasible OCI program whose yield will differ from the maximum OCI yield by as small an amount as it is wished to stipulate. Since planning according to this method takes place alternately at two levels - the centre and the sectors - organically inter-linked with one another, continuously supplementing and correcting each other, the authors have called their procedure <u>two-level planning</u>.

Actually the inspiration for the development of the allocation technique of the OCI problem and the iterative method of solution, was derived from the present planning practice in a Socialist economy.

1. The method to be described, is in some degree an imitation of the usual course of planning. The National Planning Bureau, acting on the basis of the requirements of economic policies and of general information about the various sectors, works out a preliminary draft plan which contains general targets /quota figures/ for the sectors. The centre makes a provisional distribution of the available resources, material, manpower, etc. among the sectors, and at the same time also allocates their output targets. The sectors then proceed through their own detailed calculations, on the basis of their concrete conditions, to give "substance" to the quotas and to lend concrete meaning to the central targets. In so doing, they also make recommendations for changes to the Planning Bureau. This is what is in economic usage called "counter-planning". On the basis of the counter-plans the National Planning Bureau modifies its original targets and again sends them down to the sectors. The proposed method is an attempt to aid this process of

"planning and counter-planning" by means of objective criteria.

2. The procedure recommended also simulates the usual practice of planning in another respect. It repeatedly happens that the centre gives the sectors certain directives and asks them to report with what degree of economic efficiency the task can be carried out. The sectors express the efficiency of their activities through various "indices of economic efficiency", whose structure is prescribed by the centre. The method to be treated incorporates this "reporting back" process in a unified system, where the sectors at each step report back one type of economic efficiency index - the shadow prices derived from programming - to the centre, for the evaluation of the directives obtained from there.

3. Mention has been made of the fact that in some sectors of industry mathematical programming methods are also being used to elaborate plans on the sector scale. In these programming models certain directives received from the National Planning Bureau - the output targets, manpower limitations, etc. - figure as constants on the right hand side of the constraints. These programs indeed suggest that it ought to be worth while to compare the results of the sector programs and utilize them to improve the directives and wuotas derived from the national plan. The function of the proposed process is to lend an organized form to such comparisons and the macroeconomic plan corrections taking place on their basis, in fact organically to link up the programming work done at the sector levels.

The first part of this paper discusses a general model, within whose scope the symbols and definitions may be more easily presented, and the mathematical theorems more easily proved. Section 1.1 treates the transformation of the OCI program into a two-level one, while section 1.2 expounds the latter's transformation into a polyhedral game, and its iterative solution[4]. These sections permit an insight into how the method may be used for the solution of general linear programming problems - some questions which arise, and an example, are dealt with briefly in section 1.3. In the second part of the paper, the results of the first are applied to the concrete model mentioned in the introduction, i.e. to the problem of long-term macroeconomic planning. Section 2.1 is a description of the model, section 2.2 presents the process of iteration, while section 2.3 discusses some problems of economics arising in connection with the model.

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[4] The symbols used in these sections are as follows: 1. Greek letters denote real numbers, i, n and N are positive whole numbers. 2. Small Latin letters /except i and n / mean vectors, capital Latin letters /except N /, matrices. The dash / ' / is used for transposition. A vector /if not qualified/ is understood to mean a column vector /a one-column matrix/. Row vectors are indicated by the dash put after a column vector. 3. Written characters in capitals are used to indicate sets.

1. THE GENERAL MODEL

1.1. The transformation of the overall central information problem into a two-level problem

Let

/1.1/ $Ax \leqq b\,,\ x \geqq 0\,,\ c'x \to \max!$ resp. $y'A \geqq c'\,,\ y \geqq 0\,,\ y'b \to \min!$

be the canonical forms[5] of the primal and dual versions respectively, in the OCI problem of the general model. The primal variable of the OCI problem /the vector x / is called the OCI program, the dual variable /the vector y / the OCI shadow price system. Let $\mathfrak{X}$ denote the set of feasible OCI programs, and $\mathfrak{X}^*$ the set of optimal OCI programs, moreover let $\mathcal{Y}$ be the set of feasible OCI shadow price systems and $\mathcal{Y}^*$ the set of optimal OCI shadow price systems[6], i.e.

/1.2/ $\mathfrak{X} = \{x : Ax \leqq b, x \geqq 0\}\,,\quad \mathfrak{X}^* = \{x^* : x^* \in \mathfrak{X}\,, c'x^* = \max_{x \in \mathfrak{X}} c'x\}\,,$

/1.3/ $\mathcal{Y} = \{y : y'A \geqq c', y \geqq 0\}\,,\quad \mathcal{Y}^* = \{y^* : y^* \in \mathcal{Y}\,, y^{*\prime}b = \min_{y \in \mathcal{Y}} y'b\}\,.$

Let is be assumed that the OCI problem is solvable, i.e. that

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5 The primal-dual versions of all linear programming problems can be transformed into the symmetrical form /1.1/.

6 In the case of elements z of arbitrary nature $\{z : \quad\}$ denotes the set of those elements z which satisfy the condition following after the colon.

an optimal OCI program exists; $\mathcal{X}^* \neq \emptyset$.[7] It is known[8] that there then also exists an optimal OCI shadow price system: $\mathcal{Y}^* \neq \emptyset$, moreover, the maximum value of the objective function in the primal version and the minimum value of the objective function in the dual version are equal - their common value is the optimum ϕ of the OCI problem:

/1.4/ $$\max_{x \in \mathcal{X}} c'x = \min_{y \in \mathcal{Y}} y'b = c'x^* = y^{*\prime}b = \phi \qquad (x^* \in \mathcal{X}^*,\ y^* \in \mathcal{Y}^*).$$

The solvability of the OCI problem is incidentally equivalent to the assumption that there exists a feasible OCI program and also a feasible OCI shadow price system[9], i.e.

/1.5/ $$\mathcal{X} \neq \emptyset \quad \text{and} \quad \mathcal{Y} \neq \emptyset .$$

Let

/1.6/ $$A = [A_1, \dots, A_n], \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad c' = [c'_1, \dots, c'_n]$$

be the mutually corresponding partitioning of the matrix A, the OCI program x and the OCI objective function-vector c' in the primal version of the OCI problem. Then in place of /1.1/ the equivalent forms

7 $\emptyset$ is the symbol of the empty set.

8 Goldman and Tucker [8] , Corollary 1A, p.60.

9 Goldman and Tucker [8] , Theorem 2, p.61.

$$\text{/1.7/}\quad \left\{\begin{array}{l} A_1x_1+\dots+A_nx_n \leqq b \\ x_1 \geqq 0 \\ \quad\ddots \\ x_n \geqq 0 \\ \\ c_1'x_1+\dots+c_n'x_n \rightarrow \max! \end{array}\right\} \text{ and } \left\{\begin{array}{l} y'A_1 \geqq c_1' \\ \quad\ddots \\ y'A_n \geqq c_n' \\ y \geqq 0 \\ y'b \rightarrow \min! \end{array}\right.$$

may be used.

If the sum of the vectors $u_1, \dots, u_n$ /of the same size as the bounding vector b/ is itself b, i.e. if they satisfy the <u>bounding vector partitioning condition</u>

$$\text{/1.8/}\qquad u_1+\dots+u_n = b$$

then the vector

$$\text{/1.9/}\qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

composed of them is called a <u>central program</u>, the vector u_i is the <u>i-th sector component</u> of the central program u, while the <u>i-th sector problem under the central program</u> u /or, under the sector component u_i/ is understood to mean the linear programming problem

$$\text{/1.10/}\quad A_ix_i \leqq u_i,\ x_i \geqq 0,\ c_i'x_i \rightarrow \max! \quad\text{and}\quad y_i'A_i \geqq c_i',\ y \geqq 0,\ y'b \rightarrow \min!$$

$(i = 1, \dots, n)$. In /1.10/ x_i is the <u>i-th sector program</u>, while y_i is the <u>i-th sector shadow price system</u>. In the i-th sector problem under the sector component u_i, let $\mathfrak{X}_i(u_i)$ stand for the set of <u>feasible sector programs</u>, $\mathfrak{X}_i^*(u_i)$ for the set of <u>optimal sector programs</u>, moreover $\mathfrak{Y}_i$ for the set of <u>feasible sector shadow price systems</u>, and $\mathfrak{Y}_i^*(u_i)$

for the set of <u>optimal sector shadow price systems</u>, i.e.:

$$/1.11/\quad \begin{cases} \mathfrak{X}_i(u_i) = \{x_i : A_i x_i \leqq u_i \, , \; x_i \geqq 0\} \, , \\ \mathfrak{X}_i^*(u_i) = \{x_i^* : x_i^* \in \mathfrak{X}_i(u_i) \, , \, c_i x_i^* = \max\limits_{x_i \in \mathfrak{X}_i(u_i)} c_i' x_i\} \end{cases}$$

$$/1.12/\quad \begin{cases} \mathcal{Y}_i \quad = \{y_i : y_i' A_i \geqq c_i' \, , \; y_i \geqq 0\} \, , \\ \mathcal{Y}_i^*(u_i) = \{y_i^* : y_i^* \in \mathcal{Y}_i \quad , \quad y_i^* u_i = \min\limits_{y_i \in \mathcal{Y}_i} y_i' u_i\} \end{cases}$$

$$(i = 1, \dots, n) \; .$$

Let us find the condition for the solvability of all the sector problems. Since it follows[10] from /1.12/ and /1.3/ that $\mathcal{Y} = \mathcal{Y}_1 \cap \dots \cap \mathcal{Y}_n$, and because according to the assumption made with regard to the solvability of the OCI problem /1.5/ states that $\mathcal{Y} \neq \mathcal{O}$, therefore

$$/1.13/ \qquad \mathcal{Y}_i \neq \mathcal{O} \qquad\qquad i = 1, \dots, n \, .$$

Hence two necessary and sufficient conditions may be deduced for the solvability of the i-th sector programming problem under u_i. The first, from the theorem quoted in [9], is that

$$/1.14/ \qquad \mathfrak{X}_i(u_i) \neq \mathcal{O} \, .$$

The second is that $y_i' u_i$ is bounded from below[11] on the set $\mathcal{Y}_i$. This latter statement is best put in another form. Let

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10 $\mathcal{A} \cap \mathcal{B} \cap \dots$ is the intersection of the sets $\mathcal{A}, \mathcal{B}, \dots$

11 Goldman and Tucker [8] , Corollary 1B, p.60.

/1.15/ $\mathcal{Y}_i = Y_i^\Delta + \bar{Y}_i^< = \{Y_i q_i + \mu_i \bar{Y}_i \bar{q}_i : 1'q_i = 1'\bar{q}_i = 1,\ q_i \geqq 0,\ \bar{q}_i \geqq 0,\ \mu_i \geqq 0\}$

be the canonical decomposition[12] of the polyhedral set $\mathcal{Y}_i$. Then the boundedness condition of $y_i' u_i$ on the set $\mathcal{Y}_i$ may be written in the form

/1.16 $$\bar{Y}_i' u_i \geqq 0 .$$

Let those central programs for which all the sector problems are solvable, be called evaluable central programs. According to /1.8/ and /1.16/ the set $\dot{\mathcal{U}}$ of the evaluable central programs can be written in the form as follows:

/1.17/ $$\dot{\mathcal{U}} = \left\{ \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} : u_1 + \dots + u_n = b,\ \bar{Y}_1' u_1 \geqq 0, \dots, \bar{Y}_n' u_n \geqq 0 \right\} .$$

$\dot{\mathcal{U}}$ is therefore a polyhedral convex set.

Let $\mathcal{X}(u)$ denote those OCI programs which may be composed from the sector programs feasible under the evaluable central

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12 Goldman [7] , pp.44-49. 1' denotes a row vector whose every component is 1. In /1.15/ therefore, q_i and $\bar{q}_i$ are so-called probability vectors; non-negative vectors, the sum of components of which equals 1. Y_i is the matrix consisting of the extreme points of $\mathcal{Y}_i$. If $\mathcal{Y}_i$ is bounded, then by definition $\bar{Y}_i = 0$. Otherwise, $\bar{Y}_i$ is the matrix consisting of the extreme vectors of the set $\bar{\mathcal{Y}}_i$, which consists of the probability vectors satisfying the reduced homogeneous system $y_i' A_i \geqq 0,\ y_i \geqq 0$.

program $u = [u_1', \ldots, u_n']'$. Then[13]

$$/1.18/ \quad \mathcal{X}(u) = \mathcal{X}_1(u_1) \times \cdots \times \mathcal{X}_n(u_n) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1 \in \mathcal{X}_1(u_1), \ldots, x_n \in \mathcal{X}_n(u_n) \right\} .$$

The /proper or improper/ subset $\mathcal{U}$ of the set $\dot{\mathcal{U}}$ consisting of all the evaluable central programs, is said to <u>generate</u> $\mathcal{X}$, or in other words $\mathcal{U}$ is the <u>generating set of</u> $\mathcal{X}$, if[14]

$$/1.19/ \qquad \mathcal{X} = \bigcup_{u \in \mathcal{U}} \mathcal{X}(u) .$$

For the case of $\mathcal{U} = \dot{\mathcal{U}}$ /1.19/ is valid, i.e. the non-void polyhedral convex set $\dot{\mathcal{U}}$ consisting of all the evaluable central programs, generates the set $\mathcal{X}$ of the feasible OCI programs. This may be proved in two steps. 1. From /1.5/ $\mathcal{X} \neq \emptyset$. Let $x = [x_1', \ldots, x_n']' \in \mathcal{X}$, and let the components $u_1 = A_1 x_1, \ldots, u_{n-1} = A_{n-1} x_{n-1}$, $u_n = b - (u_1 + \cdots + u_{n-1})$ be defined. Obviously $x_i \in \mathcal{X}_i(u_i)$, so that $\mathcal{X}_i(u_i) \neq \emptyset$ $(i = 1, \ldots, n)$, and consequently, because of /1.14/ $u = [u_1', \ldots, u_n']'$ is an evaluable central program, so that $\dot{\mathcal{U}} \neq \emptyset$. $\dot{\mathcal{U}}$ is therefore a <u>non-void</u> set. Moreover $x \in \mathcal{X}(u)$ and since the above construction provides such a central program

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13 $\mathcal{A} \times \mathcal{B} \times \ldots$ is the direct product of the sets $\mathcal{A}, \mathcal{B}, \ldots$ If $\mathcal{A}, \mathcal{B}, \ldots$ are sets in column vector spaces, then the general element of the direct product set is the column vector composed in the manner shown by /1.18/.

14 $\bigcup$ is the symbol for the union of sets.

$u \in \dot{\mathcal{U}}$ for each OCI program $x \in \mathcal{X}$, therefore[15] $\mathcal{X} \subset \bigcup_{u \in \dot{\mathcal{U}}} \mathcal{X}(u)$. 2. If $u = [u'_1, \ldots, u'_n]' \in \dot{\mathcal{U}}$ and $x_i \in \mathcal{X}_i(u_i)$ i.e. $A_i x_i \leqq u_i$, $x_i \geqq 0$ $(i = 1, \ldots, n)$, then because of the bounding vector partitioning condition /1.8/, $A_1 x_1 + \ldots + A_n x_n \leqq u_1 + \ldots + u_n = b$, $x = [x'_1, \ldots, x'_n]' \geqq 0$, so that $x \in \mathcal{X}$. Hence $\mathcal{X}(u) \subset \mathcal{X}$, $\bigcup_{u \in \mathcal{U}} \mathcal{X}(u) \subset \mathcal{X}$.

The generating property of /1.19/ may also be possessed by a non-void <u>proper</u> polyhedral subset of the set $\dot{\mathcal{U}}$. By way of an example actually used in the model, consider the following: In the case of a matrix A of special form, /1.7/ may assume the form

/1.20/ $$A^{\#}_1 x_1 + \cdots + A^{\#}_n x_n \leqq b^{\#}$$

/1.21/ $$\begin{cases} A^{o}_1 x_1 & \leqq b^{o}_1 \\ \quad \ddots & \\ \qquad A^{o}_n x_n & \leqq b^{o}_n \end{cases}$$

/1.22/ $$x_1 \geqq 0, \ldots, x_n \geqq 0$$

/1.23/ $$c'_1 x_1 + \ldots + c'_n x_n \longrightarrow \max!$$

/Here $A^{o}_i x_i \leqq b^{o}_i$ comprises those conditions of the OCI problem which only refer to the i-th sector. These conditions may be called the <u>special sector conditions</u> of the i-th sector./ It is useful here also to put the central program in the form

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15 $\mathcal{A} \subset \mathcal{B}$ means that $\mathcal{A}$ is a subset of $\mathcal{B}$

/1.24/ $$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \qquad u_i = \begin{bmatrix} u_i^{\#} \\ u_{i1}^{o} \\ \vdots \\ u_{in}^{o} \end{bmatrix} \qquad i = 1, \dots, n$$

and in this case the bounding vector partitioning condition is expressed by the equations

/1.25/ $$u_1^{\#} + \dots + u_n^{\#} = b^{\#}$$

/1.26/ $$u_{1i}^{o} + \dots + u_{ni}^{o} = b_i^{o} \qquad i = 1, \dots, n .$$

Let $\dot{\mathcal{U}}$ here again stand for the set of all the evaluable central programs, and $\mathcal{U}$ here denote that subset of $\dot{\mathcal{U}}$, in which each sector "obtains" in full the bounds occurring in its special sector conditions, that is,

/1.27/ $$\mathcal{U} = \left\{ u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} : u \in \dot{\mathcal{U}}, \quad u_1 = \begin{bmatrix} u_1^{\#} \\ b_1^{o} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, u_n = \begin{bmatrix} u_n^{\#} \\ 0 \\ \vdots \\ 0 \\ b_n^{o} \end{bmatrix} \right\} .$$

It is obvious that $\mathcal{U}$ is a non-void convex polyhedral subset of $\dot{\mathcal{U}}$, which does not necessarily comprise every evaluable central program[16].

Returning to the general case, let $\mathcal{U}$ now be such a non-void convex polyhedral subset of $\dot{\mathcal{U}}$, that generates $\mathfrak{X}$. Let us fix $\mathcal{U}$ and let its elements be called <u>feasible central programs</u>. For any feasible central program $u = [u_1', \dots, u_n']'$,

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16 E.g. for the case of $b^{\#} \geqq 0$, $b_1^{o} > 0$, $\mathcal{U}$ is a proper subset of $\dot{\mathcal{U}}$.

the sector-optima $\varphi_i(u_i) = \max\limits_{x_i \in \mathcal{X}_i(u_i)} c_i' x_i = \min\limits_{y_i \in \mathcal{Y}_i} y_i' u_i$; $(i = 1, \ldots, n)$, and their sum, the overall optimum $\varphi(u) = \varphi_1(u_1) + \cdots + \varphi_n(u_n)$ under u may be defined. These are continuous and domain by domain linear concave functions of u. For if $y_{i1}, \ldots, y_{iN_i}$ denote the extreme points of $\mathcal{Y}_i$, i.e. if $Y_i = [y_{i1}, \ldots, y_{iN_i}]$ may be written in /1.15/, then because u is evaluable, $\varphi_i(u_i) = $ $= \min\limits_{y_i \in \mathcal{Y}_i} y_i' u_i = \min\{y_{i1}' u_i, \ldots, y_{iN_i}' u_i\}$, so that it is the lower envelope of a finite number of linear functions $(i = 1, \ldots, n)$.

The two-level problem obtained from the OCI problem by means of the sector decomposition /1.6/ and choice of the set of feasible central programs as above, is understood to mean a problem as follows.

/1/ At the "central level", to determine the feasible central program/s/ which yield/s/ the maximal overall optimum, in other words to solve the concave programming problem $u \in \mathcal{U}, \varphi(u) \longrightarrow \max!$ and to determine the set

/1.28/ $$\mathcal{U}^* = \{u^* : \varphi(u^*) = \max_{u \in \mathcal{U}} \varphi(u)\}$$

consisting of the optimal central programs.

/2/ At the "sector level", in each sector to determine the optimal sector program/s/ belonging to the optimal central program component/s/, i.e. for each $u^* = [u_1^{*\prime}, \ldots, u_n^{*\prime}]' \in \mathcal{U}^*$ to solve the sector problems $A_i x_i \leqq u_i^*$, $x_i \geqq 0$, $c_i' x_i \longrightarrow \max!$ thus to determine the sets $\mathcal{X}_i^*(u_i^*)$ for $i = 1, \ldots, n$, defined in /1.11/.

/3/ To compose the OCI program/s/ which may be obtained from the optimal sector programs under the optimal central program/s/, in other words to determine the union of the sets

$$/1.29/ \quad \mathfrak{X}^*(u^*) = \mathfrak{X}_1^*(u_1^*) \times \dots \times \mathfrak{X}_n^*(u_n^*) = \left\{ \begin{bmatrix} x_1^* \\ \vdots \\ x_n^* \end{bmatrix} : x_1^* \in \mathfrak{X}_1^*(u_1^*), \dots, x_n^* \in \mathfrak{X}_n^*(u_n^*) \right\}$$

in the form $\bigcup_{u^* \in \mathcal{U}^*} \mathfrak{X}^*(u^*)$.

THEOREM 1. <u>Any two-level problem derived from a solvable OCI problem is itself also solvable, and its solution is equivalent to the solution of the OCI problem:</u>

$$/1.30/ \quad \mathcal{U}^* \neq \emptyset \qquad \text{and} \qquad \mathfrak{X}^* = \bigcup_{u^* \in \mathcal{U}^*} \mathfrak{X}^*(u^*) .$$

<u>The maximum value of the overall optimum is equal to the optimum of the OCI problem:</u>

$$/1.31/ \quad \max_{u \in \mathcal{U}} \varphi(u) = \varphi(u^*) = \max_{x \in \mathfrak{X}} c'x = \phi \qquad (u^* \in \mathcal{U}^*) .$$

<u>Proof</u>. The statements of the theorem may be read from the following:

$$\phi = \max_{x \in \mathfrak{X}} c'x = \max_{x \in \bigcup_{u \in \mathcal{U}} \mathfrak{X}(u)} c'x = \max_{u \in \mathcal{U}} \left(\max_{x \in \mathfrak{X}(u)} c'x \right) = \max_{u \in \mathcal{U}} \left(\sum_{i=1}^{n} \max_{x_i \in \mathfrak{X}_i(u_i)} c_i' x_i \right) =$$

$$= \max_{u \in \mathcal{U}} \varphi(u) = \varphi(u^*) .$$

1.2. The transformation of the two-level problem into a polyhedral game, and iterative solution of the latter

The objective function of the concave programming problem to be solved at the "central level" of the two-level problem is the overall optimum $\varphi(u)$. This function may indeed be determined on the basis of the data of the OCI problem and its decomposition into sectors, but it is not an easy task. The two-level problem is therefore suitably transformed. Let $\mathcal{V}$ denote the set of feasible sector shadow price system teams according to /1.12/:

$$/1.32/ \quad \mathcal{V} = \mathcal{Y}_1 \times \dots \times \mathcal{Y}_n = \left\{ v = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} : y_1 \in \mathcal{Y}_1, \dots, y_n \in \mathcal{Y}_n \right\} .$$

Then the following expression is obtained for the overall optimum:

$$/1.33/ \quad \varphi(u) = \sum_{i=1}^{n} \varphi_i(u_i) = \sum_{i=1}^{n} \min_{y_i \in \mathcal{Y}_i} y_i' u_i = \min_{\substack{y_i \in \mathcal{Y}_i \\ i=1,\dots,n}} \sum_{i=1}^{n} y_i' u_i = \min_{v \in \mathcal{V}} v' u \; .$$

Hence, according to /1.31/, the following may be written for the OCI optimum:

$$/1.34/ \qquad \phi = \max_{u \in \mathcal{U}} \min_{v \in \mathcal{V}} v' u \; .$$

Let us define a polyhedral game[17] in the following terms: Let $\mathcal{U}$ be the set of strategies of the maximizing, and $\mathcal{V}$ of the mini-

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17 See Wolfe [25] . In this case $m = n$, $X = \mathcal{U}$, $Y = \mathcal{V}$, $A = E$ = identity matrix.

mizing player, and the homogeneous bilinear function $v'u$ $(u \in \mathcal{U}, v \in \mathcal{V})$ be the pay-off function of the game. The maximizing player may be identified with the "centre", the minimizing player with the "team of sectors".[18] Consequently we may speak of a <u>central strategy</u> on place of the feasible central program, <u>sector strategy</u> in place of the feasible sector shadow price system team, and in the case of both strategies we may consider the various <u>sector components</u> of the strategy. The game which has thus been defined is called the <u>polyhedral game derived from the two-level problem</u> or <u>from the OCI problem</u> through the given decomposition of sectors and the given choice of the set of feasible central programs, in symbols: $(\mathcal{U}, \mathcal{V})$. What the relation /1.34/ expresses, is that the optimum of the OCI problem is the max-min /lower/ value of the polyhedral game derived from it. The connection between the OCI problem, moreover the two-level problem and the polyhedral game derived from it, is contained in the following theorem:

THEOREM 2. <u>A polyhedral game derived from a solvable OCI problem is itself solvable and its value equals the OCI optimum. The optimal strategies of the "central player" are the optimal central programs appearing in the corresponding two-level problem. Among the optimal strategies of the "sector-team player" there is always a strategy whose sector components are equal - the necessary and sufficient condition for a sector strategy whose sector components are equal to be optimal, is that it should be the optimal counter-strategy</u>

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18 See the definition of a "person", e.g. in McKinsey's book [20], p.4, para. 3.

against some central strategy[19]. In an optimal sector strategy whose sector components are equal, this common sector component forms an optimal OCI shadow price system and vice versa.

Proof. 1/ In the case of a solvable OCI problem the OCI optimum ϕ exists and is finite, while according to /1.34/ it is equal to the max-min value of the polyhedral game $(\mathcal{U},\mathcal{V})$. From this it follows according to the theorem of Wolfe [25], that ϕ is at the same time also the min-max /upper/ value of this game, so that $(\mathcal{U},\mathcal{V})$ is solvable, and its value is ϕ, and both players have optimal strategies.

2/ Since according to /1.33/ $\varphi(u)$ is the minimum of the payoff function $v'u$ on the set $\mathcal{V}$, the set consisting of the optimal strategies of the central player equals the set in /1.28/.

3/ Using the notation of /1.3/-/1.12/ it will first be shown that

/1.35/ $$\mathcal{Y}_1^*(u_1)\cap\dots\cap\mathcal{Y}_n^*(u_n) = \begin{cases} \mathcal{Y}^*, & \text{if} \quad u\in\mathcal{U}^* \\ \emptyset, & \text{if} \quad u\notin\mathcal{U}^* \end{cases}$$

so that an optimal counter-strategy the sector-components of which are equal exists only as a counter-strategy against some optimal central strategies, but then always exists, and the common sector component is an optimal OCI shadow price system. As the first part of the proof of /1.35/ it can be demonstrated that for $y^*\in\mathcal{Y}^*$, $u^*\in\mathcal{U}^*$ it is true that $y^*\in\mathcal{Y}_i^*(u_i^*)$ $(i=1,\dots,n)$. For if

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19 $\hat{v}\in\mathcal{V}$ is an optimal counter-strategy against the central strategy $u\in\mathcal{U}$, if $\hat{v}'u = \min_{v\in\mathcal{V}} v'u = \varphi(u)$.

the opposite were the case, then we would have $\phi = \varphi(u^*) =$

$$= \sum_{i=1}^{n} \varphi_i(u_i^*) = \sum_{i=1}^{n} \min_{y_i \in \mathcal{Y}_i} y_i' u_i^* < \sum_{i=1}^{n} y^{*\prime} u_i^* = y^{*\prime} \sum_{i=1}^{n} u_i^* = y^{*\prime} b = \phi$$

which is impossible. As the second part of the proof of /1.35/ it can be shown that in the reverse case, for $\hat{y} \in \mathcal{Y}_i^*(\hat{u}_i)$ $(i=1,\dots,n)$ it is true that $\hat{y} \in \mathcal{Y}^*$, $\hat{u} \in \mathcal{U}^*$. For let $y \in \mathcal{Y}$; then

$$y'b = y' \sum_{i=1}^{n} \hat{u}_i = \sum_{i=1}^{n} y'\hat{u}_i \geq \sum_{i=1}^{n} \min_{y_i \in \mathcal{Y}_i} y_i' \hat{u}_i = \sum_{i=1}^{n} \varphi_i(\hat{u}_i) = \varphi(\hat{u}_i) =$$

$$= \sum_{i=1}^{n} \hat{y}'\hat{u}_i = \hat{y}' \sum_{i=1}^{n} \hat{u}_i = \hat{y}'b, \text{ so that } y'b \geq \hat{y}'b \text{ i.e. } \hat{y} \in \mathcal{Y}^*.$$

It follows furthermore, that $\varphi(\hat{u}) = \hat{y}'b = \phi$, so that $\hat{u} \in \mathcal{U}^*$. /1.35/ has thus been proved. To complete the proof of Theorem 2 it is now only necessary to show that for any optimal OCI shadow price system y^*, the sector strategy $\tilde{v}^* = [y^{*\prime}, \dots, y^{*\prime}]'$ with equal sector components y^* is an optimal one. For this it is sufficient to demonstrate that $\tilde{v}^*$, together with any optimal central strategy $u^* \in \mathcal{U}^*$, forms a saddle-point of the function $v'u$ in $(\mathcal{U}, \mathcal{V})$. For let $u^* = [u_1^{*\prime}, \dots, u_n^{*\prime}]'$ be the chosen optimal central strategy, and $u = [u_1', \dots, u_n']' \in \mathcal{U}$ as well as $v = [y_1', \dots, y_n']' \in \mathcal{V}$ be arbitrary. Then

$$\tilde{v}^{*\prime} u = \sum_{i=1}^{n} y^{*\prime} u_i = y^{*\prime} \sum_{i=1}^{n} u_i = y^{*\prime} b = \phi = y^{*\prime} \sum_{i=1}^{n} u_i^* = \sum_{i=1}^{n} y^{*\prime} u_i^* = \tilde{v}^{*\prime} u^* \leq$$

$$\leq \sum_{i=1}^{n} y_i' u_i^* = v'u^*, \text{ so that } \tilde{v}^{*\prime} u \leq \tilde{v}^{*\prime} u^* \leq v'u^* \quad (u \in \mathcal{U}, v \in \mathcal{V}).$$

Theorem 2. is thus proved.

The OCI problem has thus been reduced to a solution of the polyhedral game $(\mathcal{U},\mathcal{V})$ derived from it. A method was now sought for its solution, which would utilize the decomposition of the problem, and be built up of partial calculations that can take place separately at the centre and in the sectors. To achieve this, let the concept of an <u>evaluable sector strategy</u> /or an <u>evaluable sector shadow price system team</u>/ be defined. They are understood to mean a sector strategy $v \in \mathcal{V}$, against which there exists an optimal central counter-strategy; in other words one where the linear programming $u \in \mathcal{U}, v'u \rightarrow max!$ is solvable. Let a derived polyhedral game $(\mathcal{U},\mathcal{V})$ be called <u>regular</u> if all its sector strategies are evaluable. <u>It is henceforth assumed that the polyhedral game $(\mathcal{U},\mathcal{V})$ is regular</u>. Consider that according to the definition of a two-level problem every element of $\mathcal{U}$ an evaluable central strategy, i.e. a central program u for which the linear programming $v \in \mathcal{V}$, $v'u \longrightarrow min!$ is solvable. In the case of a regular polyhedral game therefore, there is an optimal counter-strategy against each strategy of both players. It hence follows that all regular polyhedral games are strategically reducible[20] to a matrix game[21]. For let $\mathcal{U} = U^{\Delta} + \bar{U}^{<}$, and $\mathcal{V} = V^{\Delta} + \bar{V}^{<}$ be the canonical decomposition of the strategy sets concerned. From the two kinds of solvability assumptions it immediately follows, analogously with /1.16/, that $V'\bar{U} \leqq 0$, $\bar{V}'\bar{U} = 0$ and $\bar{V}'U \geqq 0$, moreover that the strategy $u = Up + \lambda\bar{U}\bar{p} \in \mathcal{U}$ may /in the wide sense/ be dominated by the strategy $u^{\Delta} = Up \in U^{\Delta}$, the strategy $v = Vq + \mu\bar{V}q \in \mathcal{V}$ by the

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[20] A game is said to be <u>strategically reducible</u> to another game, if the latter is solvable and its every solution /optimal strategy-pair/ is a solution of the original game, too.

[21] For a definition of the matrix game see e.g. p. 17 of Karlin's work [9].

strategy $v^{\Delta} = Vq \in V^{\Delta}$, i.e. that for all strategies $v^{o} \in \mathcal{V}$ $v^{o\prime}u \leqq v^{o\prime}u^{\Delta}$, and for all strategies $u^{o} \in \mathcal{U}$ $v'u^{o} \geqq v^{\Delta\prime}u^{o}$, where $p, \bar{p}, q, \bar{q}$ are probability vectors, λ and μ are non-negative numbers. The polyhedral game $(\mathcal{U}, \mathcal{V})$ is therefore strategically reducible to the polyhedral game (U^{Δ}, V^{Δ}), and this in turn is isomorphic with the matrix game having a pay-off matrix $V'U$. It should be pointed out that this matrix game is only stated in an implicit form, since the pay-off matrix $V'U$ is not directly known, the only available fact being that U and V are the matrices composed by the extreme vectors of the polyhedral sets $\mathcal{U}$ and $\mathcal{V}$, defined by the inequality systems of the centre and the sectors respectively. If, therefore, it is intended to find a solution of the regular polyhedral game $(\mathcal{U}, \mathcal{V})$ by solving the matrix game with the pay-off matrix $V'U$, then for this reason itself - apart from the difficulties of a computing problem of at least the same size as the original OCI problem - it is impossible to apply direct calculation procedures. It is possible, on the other hand, to use the <u>fictitious play</u> method of Brown and Robinson.[22] According to the above, it is possible against each central strategy $u \in \mathcal{U}$ of the regular polyhedral game $(\mathcal{U}, \mathcal{V})$, to state an optimal counter-strategy $v^{*}(u) \in V^{\Delta}$ while against each sector strategy $v \in \mathcal{V}$ it is possible to state an optimal counter-strategy $u^{*}(v) \in U^{\Delta}$, for which

/1.36/ $$v^{*}(u)'u = \min_{v \in \mathcal{V}} v'u \qquad v'u^{*}(v) = \max_{u \in \mathcal{U}} v'u$$

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22 Brown [1], [2]; Robinson [21]. For a detailed discussion: Karlin [9], pp. 179-189.

holds. $v^*(u)$ is called the <u>regular valuation</u> of u, $u^*(v)$ of v. The <u>regular fictitious play</u> of the regular polyhedral game (U, V) is understood to mean the construction according to the rule stated below, of the strategy series $u^*\langle 1\rangle, u^*\langle 2\rangle, \ldots$ $\ldots, u^*\langle N\rangle, \ldots$ within U^Δ and $v^*\langle 1\rangle, v^*\langle 2\rangle, \ldots, v^*\langle N\rangle, \ldots$ within V^Δ.

<u>Phase 1</u>. Step I. Select any central strategy $u^{(1)} \in U^\Delta$. Step II. By definition $u^*\langle 1\rangle = u^{(1)}$. Step III. /The regular valuation of $u^*\langle 1\rangle$ /: The determination of $v^{(1)} = v^*(u^*\langle 1\rangle)$. Step IV. By definition $v^*\langle 1\rangle = v^{(1)}$.

Next the process goes on for phases 2, 3, ...

<u>Phase N</u>. $(N = 2,3,\ldots)$ Step I. /The regular valuation of $v^*\langle N-1\rangle$ / : The determination of $u^{(N)} = u^*(v^*\langle N-1\rangle)$. Step II. /"Mixing" with the term of the previous phase/: The calculation of $u^*\langle N\rangle = \frac{N-1}{N} u^*\langle N-1\rangle + \frac{1}{N} u^{(N)}$. Step III. /The regular valuation of $u^*\langle N\rangle$ /: The determination of $v^{(N)} = v^*(u^*\langle N\rangle)$. Step IV. /"Mixing" with the term of the previous phase/: The calculation of $v^*\langle N\rangle = \frac{N-1}{N} v^*\langle N-1\rangle + \frac{1}{N} v^{(N)}$.

As $\phi = \max\limits_{u \in U} \min\limits_{v \in V} v'u = \min\limits_{v \in V} \max\limits_{u \in U} v'u$, it may easily be deduced from the definitions of the valuations that the <u>upper optimum</u> $\phi^*\langle N\rangle = \max\limits_{u \in U} v^*\langle N-1\rangle' u = \max\limits_{u \in U^\Delta} v^*\langle N-1\rangle' u =$ $= v^*\langle N-1\rangle' u^{(N)}$ in the N-th phase $(N = 2,3,\ldots)$ supplies an upper estimate, the <u>lower optimum</u> $\varphi^*\langle N\rangle = \min\limits_{v \in V} v' u^*\langle N\rangle = \min\limits_{v \in V^\Delta} v' u^*\langle N\rangle =$ $= v^{(N)\prime} u^*\langle N\rangle$ in the N-th phase $(N = 1,2,3,\ldots)$ a lower estimate for the OCI optimum ϕ, i.e.

/1.37/ $\phi^*\langle N\rangle \geqq \phi \quad (N = 2,3,\ldots) \qquad \varphi^*\langle N\rangle \leqq \phi \quad (N = 1,2,3,\ldots).$

Since, moreover, the series also imply, on account of their construction, the fictitious play of the matrix game with the pay-off matrix $V'U$, the Brown-Robinson theorem is valid, so that

/1.38/ $$\lim_{N\to\infty} \phi^*\langle N\rangle = \lim_{N\to\infty} \varphi^*\langle N\rangle = \phi$$

and the limit points of the series $\{u^*\langle N\rangle\}$ and $\{v^*\langle N\rangle\}$ are optimal central and sector strategies.

The δ - termination of the iteration of regular fictitious play /where δ is an arbitrary small positive number/ is understood to mean the following stopping of the above construction. Let N_δ be the least positive whole number for which

/1.39/ $\phi^*\langle N_\delta\rangle - \varphi^*\langle N_\delta\rangle \leqq \delta$ or $\phi^*\langle N_\delta + 1\rangle - \varphi^*\langle N_\delta\rangle \leqq \delta$

holds. According to /1.37/ and /1.38/, N_δ may be defined for an arbitrarily small positive number δ. Then: 1. The iteration is terminated at Step II of phase N_δ, or Step I of phase $(N_\delta + 1)$, /according to whether the first or the second inequality in /1.39/ has been satisfied/. 2. The linear programming problems

/1.40/ $A_i x_i \leqq u_i^*\langle N_\delta\rangle, \; x_i \geqq 0, \; c_i' x_i \to max! \qquad i = 1,\ldots,n$

in the sectors are solved. 3. From the sector programs $x_i^{\delta *}(\delta)$ thus obtained, the feasible OCI program $x^{\delta *} = [x_1^{\delta *\prime}, \ldots, x_n^{\delta *\prime}]'$ is composed. Since $c'x^{\delta *} = \sum_{i=1}^{n} c_i' x_i^{\delta *} = \sum_{i=1}^{n} y_1^*(u_i^*\langle N_\delta\rangle)' u_i^*\langle N_\delta\rangle =$

$$= \upsilon^*(u^*\langle N_\delta\rangle)' u^*\langle N_\delta\rangle = \varphi^*\langle N_\delta\rangle ,$$ according to /1.39/

/1.41/ $$\phi - \delta \leqq c' x^{\delta *} \leqq \phi .$$

/1.41/ is briefly referred to as the fact that $x^{\delta *}$ is a δ -optimal OCI program.

As a supplement, take the case where the polyhedral game $(U, \mho)$ may be solved uniquely for the sector-team player. It hence follows that its reduced version, the matrix game with the pay-off matrix $V'U$, also possesses this property, so that according to the previously quoted Brown-Robinson theorem the series $\{ \upsilon^* \langle N \rangle \}$ is convergent and its limit is the unique optimal sector strategy. On the basis of part 3/ of Theorem 2, this is no other than that sector shadow price system team whose every sector component is /in consequence of the above conditions/ the unique, optimal OCI shadow price system.

The results pertaining to the fictitious play of the polyhedral game derived from the OCI problem, whose proofs have been furnished above, may be summarized as follows:

THEOREM 3. In the case of a regular polyhedral game derived from a solvable OCI problem the latter may be solved through the regular fictitious play to any required degree of accuracy, in the sense that for an arbitrarily small positive δ the δ -termination of regular fictitious play leads to a δ -optimal OCI program. If, at the same time, the derived regular polyhedral game may be uniquely solved for the sector-team player, the sector components of the mixed sector strategy series obtained in the course of regular fictitious play are equalized, i.e. they converge towards a common limit which is the optimal OCI shadow price system.

1.3. Supplementary comments

The OCI problem must be transformed into a regular polyhedral game. This is done by decomposition into sectors and the choice of a suitable generating set consisting of evaluable central programs. The question has not been discussed, of whether this transformation can be carried out for all linear programming problems - regarded as OCI problems - and that if a problem can be transformed, which of the several kinds of transformation it is best to choose. Nor has the problem been examined of whether the central programming problem in Step I of the iterative phases can be carried out from the point of view of computing technique, for in the general case the number of variables in the central programming, i.e. the number of components in the central program, is the product of the number of OCI conditions and the number of sectors. It should be pointed out that in the special case where the elements of the matrix A and the bounding vector b are non-negative numbers, then under any decomposition the evaluable central programs are the vectors whose components form a non-negative partition of b, so that the set of evaluable central programs is a non-void, bounded convex polyhedral set. Since it hence follows that all feasible sector shadow price system teams are evaluable, any decomposition and the choice of a generating set consisting of all the evaluable central programs, leads to a regular polyhedral game. Moreover the central programming in Step I of the iteration phases of the fictitious play decomposes into "microprogramming", in the course of which the full bounds of each OCI condition are partitioned to the sector which possessed the largest shadow price referring to this condition.

Similar results are also derived in the course of the transformation of the concrete model in hand, that of a long-term macroeconomic planning problem. For this reason the problem raised in the case of the general model is in this paper left open, and the necessary change will only be made on the concrete model concerned[23].

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[23] Further problems relating to the general model are treated in a paper by Lipták [18] , now under press.

2. THE CONCRETE MODEL: A LONG-TERM MACROECONOMIC PLANNING PROBLEM

2.1. Description of the model

While in the general model the point of departure was the OCI problem, in the case of the concrete model it is more evident to begin its discussion with the form subsequent to the decomposition into sectors. We shall therefore proceed straight away to write down the two-level problem derived from the OCI problem. Both the criterion /1.14/ and the simplifying remark /1.20/-/1.27/ were applied in formulating the set of central programs. The various components of the central program and the sector programs are, in agreement with their economic nature, denoted by different letters, while their sign has been chosen for ease of notation.

Planning is directed by the centre - in actual practice by the National Planning Bureau. There are altogether n sectors. Each sector is responsible for a particular group of products; in the subsequent discussion products will figure in place of product-groups, for the sake of brevity. The activities of the sector comprise not only the domestic production of the product concerned and the investments necessary for production, but also the export and import of the product. A long-term plan is to be worked out for a plan-term, consisting of altogether T periods.

It is not desired to use this model to determine all the targets of national plan. The point of departure is a national economic plan that has already been elaborated /by "traditional", non-mathematical means, checked with an input-output table/. Certain targets of this plan are adopted as constants in the programming table. In this paper they are called economic policy figures.

The centre issues three kinds of directive to the sectors:

1. The centre tells the i -th sector to provide a certain quantity of product to meet domestic requirements in the t -th period. This quantity, whose symbol is r_{it}, is called the <u>supply assignment</u> ($i = 1, \dots, n$; $t = 1, \dots, T$). The centre does not prescribe whether the required quantity should be met from domestic production or imports - this will be determined by the sector program. Furthermore it is the sector program that must determine whether the sector wishes - beyond satisfying domestic requirements - also to export.

2. The centre assigns to the i -th sector a certain quantity of the j -th product for the t -th period. This is symbolized by z_{ijt} and called the <u>materials quota</u> ($i=1,\dots,n$; $j=1,\dots,n$; $j \neq i$; $t = 1,\dots,T$. The materials quota comprises the j -th material derived both from home production and imports.

3. The centre assigns a certain complement of manspower to the i -th sector for the t -th period. This is symbolized by w_{it} and called the <u>manpower quota</u>.

<u>The directives are the variables of the central program</u>. The constants in the constraint system of this central program are economic policy figures. These are:

1. Q_{it}, which stands for the <u>external consumption</u> of the i -th product, necessary in the t -th period. This comprises consumption by individuals and public bodies, including improductive investments. On the other hand it does not include either exports or - apart from certain exceptions - productive investments. /The exceptions will be treated later./

2. R_{it}, which is the bound of the i -th supply assignment in the t -th period. There is no practical difficulty about determining the quantity which the product-volume required to

satisfy home needs is sure not to exceed.

3. W_t , which is the manpower quota available for productive work in the economy in the t -th year.

Those central programs will be considered feasible, for which

$$/2.1/ \qquad \sum_{\substack{j=1 \\ j\neq i}}^{n} z_{jit} + Q_{it} = r_{it} \leqq R_{it} \qquad i = 1,\dots,n\,; \quad t = 1,\dots,T$$

$$/2.2/ \qquad \sum_{i=1}^{n'} w_{it} = W_t \qquad t = 1,\dots,T$$

$$/2.3/ \qquad r_{it} \geqq 0,\ z_{ijt} \geqq 0,\ w_{it} \geqq 0 \qquad i = 1,\dots,n;\ j = 1,\dots,n,\ j \neq i, \quad t = 1,\dots,T.$$

The variables in the programming model of the i -th sector may be classified into several groups according to their economic nature.

1. Reproductive activities. These consist of the unchanged, continued operation of the output capacities for the i -th product which already existed at the beginning of the plan-term. Several kinds of these activities may be incorporated in the model according to their technical features /e.g. backward or advanced factories/. Let x_{ikt} denote the level of the k -th reproductive activity planned for the i -th sector in the t -th period[24]

[24] Here, and also in the case of the other variables /except for investment activities/, the unit of measurement for the level of activity is the quantity of the product stated in the natural units best suited for its measurement per unit period of true, or else in forint per unit period. It must be identical with the unit of measurement used for the i -th product in the corresponding central product balance according to /2.1/.

/ $x_{ikt} \geqq 0$, $k = repr$ [25], $t = 1, \ldots, T$ /.

2. Investment activities. This concept is to include both the establishment of new capacities and production upon these new facilities. Several types of investment activity may be incorporated in the model, on the basis of technical or economic features /e.g. the technology used, imported or domestic-made machinery, etc./. Moreover, within a particular type of investment activity /e.g. the establishment and operation of a particular plant in a certain way/, several kinds of investment activity may be distinguished according to the period in which the investment is begun. A separate investment variable will correspond to each of these alternatives. Let x_{ik} denote the level of the k -th investment activity of the i -th sector[26] ($x_{ik} \geqq 0$, $k = inv$) .

3. Export activities. Several kinds of export activities can figure in the model, according to their economic features /e.g. by markets, countries, etc./. Let x_{ikt} denote the level of the k- -th export activity in the case of the i -th product in the t- -th period ($x_{ikt} \geqq 0$, $k = exp$, $t = 1, \ldots, T$).

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25 Neither here, nor in the other groups of sector activities will the numbers of the activities be stated. Instead, a suitable abbreviation after the suffix k will indicate the character of the activity concerned; e.g. $k = repr$, $k = inv$, ... etc.

26 Since, according to the above, an investment activity refers not to particular periods but is a series of investment activities over the full plan term, the level x_{ik} is distinguished from the other variables in that it does not include the suffix t . The level of an investment activity is accordingly measured by the quantity of product which the facility that has been established produces when it is operated at full capacity, in terms of natural units or forint per unit plan period.

4. Bounded import activities. This group comprises only import activities which compete with the domestic output activities belonging to groups 1 and 2 and are able to replace the latter, moreover whose level is bounded by some external market factor. Several kinds of activity can figure in the model, according to their economic features /e.g. markets, etc./. Let x_{ikt} denote the level of the k -th bounded import activity in the case of the importation of the i -th product in the t -th period $(x_{ikt} \geqq 0\,,\ k = imp\,,\ t = 1,\dots,T)$.

5. Unbounded import activity. This is an import activity, which, similarly to the import activities of group 4, competes with domestic production, but its level is not bounded either by extraneous market factors, or by other influences. In some sectors there is justification for presuming that an unbounded import activity is a realistic proposition. In other sectors no such free, unbounded import activity actually exists. This type of variable is, however, used even in these latter sectors as an auxiliary variable. To programming procedure used will automatically eliminate these variables from the program, but the nature of the method requires that the import variable with no upper boundary be included in every sector model. Let the level of unbounded import in the i -th sector during the t -th period be denoted by x_{i0t} $(t = 1,\dots,T)$.

The conditions prescribed for the i -th sector program may be classified in one of two main groups. One group of conditions ensures that the sector should obey the directives received from the centre. The first condition is that

$$/2.4/ \quad r_{it} \leqq \sum_{\substack{k=repr,exp\\ imp,0}} f_{ikt}\, x_{ikt} + \sum_{k=inv} f_{ikt}\, x_{ik} \leqq R_{it} \qquad t = 1,\dots,T\,.$$

The <u>output coefficient</u> f_{ikt} in this condition, is as follows from the various sector activities:

<u>1</u>. For reproductive activities $f_{ikt} = 1$. <u>2</u>. For investment activities $f_{ikt} \geqq 0$, but for at least one t, $f_{ikt} = 1$. As a result of unit investment activity there will sometime, but at the latest during the last period, come to be established a capacity unit which will be able during one period to produce unit quantity of the k -th product. The preceding output on the other hand will depend on when the investment is begun, and on the amount of "turning up" required before it achieves normal operation. It is assumed that a specific time-distribution of output /and as we shall see, of expenditures/ is characteristic of the k -th investment activity. It is also assumed that the capacities created by means of the investment will, after "turning up", always be utilized to the normal extent. If, therefore f_{ikt} is for some value of t equal to 1, then it is also 1 for the periods $(t+1), (t+2)$ etc. In this sense then, this group of activities differs from the reproductive ones, of which it was not assumed that the existing old capacities must necessarily be utilized fully. <u>3</u>. In the case of export activities $f_{ikt} = -1$. <u>4</u>. and <u>5</u>. For bounded and unbounded import activities $f_{ikt} = 1$.

The next series of conditions linked to the central directives is that

$$/2.5/ \quad \sum_{\substack{k = repr, exp \\ imp, 0}} g_{ijkt} x_{ikt} + \sum_{k = inv} g_{ijkt} x_{ik} \leqq z_{ijt} \qquad \begin{array}{l} j = 1, \dots, n, \; j \neq i \\ t = 1, \dots, T. \end{array}$$

The <u>input coefficient</u> g_{ijkt} is the following for the various sector activities:

1. For reproductive activities $g_{ijkt} \geqq 0$. Production, through its technological character, either requires or does not require the j -th material. These materials requirement comprises both that of current operation, and also the materials requirements of the jobs necessary for the upkeep of the old facilities, for the general repair, replacement and renovation which are needed to ensure simple reproduction. 2. In the case of investment activities $g_{ijkt} \geqq 0$. This comprises the products /e.g. machines/ required by investment during the years when the new facility is established, and the products necessary during the years of operation both for current production and for the maintenance and replacement of the facilities that have been set up. As in the case of output, it is here again assumed that the k -th investment activity is characterized by a certain distribution of material requirements in time. 3., 4. and 5. For all foreign trade activity $g_{ijkt} = 0$.

Finally the last condition linked to the central directive is that

$$/2.6/ \quad \sum_{\substack{k = repr, exp \\ imp, 0}} h_{ikt} x_{ikt} + \sum_{k = inv} h_{ikt} x_{ik} \leqq w_{it} \qquad t = 1,\dots,T .$$

The manpower-coefficient h_{ikt} is as follows from the various groups of activity:

1. In the case of reproductive activity $h_{ikt} > 0$, for without manpower there can be no production. 2. For investment activities $h_{ikt} \geqq 0$. Before operation is begun it is 0; later it is positive - the development of its numerical value is a characteristic function of time. 3., 4. and 5. In foreign trade activities $h_{ikt} = 0$.

Beyond the conditions to secure observance of the central directives, special condition characteristic of the particular circumstances of the sector may also be stipulated. By way of example, reproductive activities might be bounded by the upper limit of existing facilities. Certain investment activities - e.g. the modernization of existing plants - might be limited. Output in some sectors might be bounded by the country's natural resources. Certain export and import activities might be limited by market factors, etc. The special conditions may be written in the following general form[27]:

$$/2.7/ \quad \sum_{t=1}^{T} \sum_{\substack{k=\mathrm{repr,exp} \\ \mathrm{imp}}} a^{o}_{iskt} x_{ikt} + \sum_{k=\mathrm{inv}} a^{o}_{isk} x_{ik} \leqq b^{o}_{is} \qquad s = \mathrm{spec.}$$

According to the artificial character of unbounded import activities, the coefficients of these variables are 0 in all the special conditions above.

The aim of the i -th sector's programming is that

$$/2.8/ \quad \sum_{t=1}^{T} \sum_{\substack{k=\mathrm{repr,exp} \\ \mathrm{imp},0}} c_{ikt} x_{ikt} + \sum_{k=\mathrm{inv}} c_{ik} x_{ik} \longrightarrow \max!$$

i.e. the maximization of the sectoral objective function on the left hand side of /2.8/. Here c_{ikt} and c_{ik} are the foreign exchange returns of the corresponding activity. These are the following for the various groups of activity:

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27 The abbreviation $s = \mathrm{spec.}$ expresses the fact that all the special conditions are enumerated in /2.7/.

1. and 2. For reproductive and investment activities the foreign currency returns are generally zero. An exception is formed by the production and investment activities, which require non-competitive imports that cannot be satisfied by home production. Non-competitive import costs are accounted as negative foreign currency returns. /In the ease of investment activities the foreign currency returns naturally comprise all the non-competitive import costs incurred throughout the plan term./ 3. The foreign currency returns of export activities are positive. 4. and 5. The foreign currency returns of import activities are negative. If unbounded imports are only a fictitious variable, they are weighted with very heavy negative foreign currency returns. It is also assumed with respect to the yield of foreign trade activities that

$$/2.9/ \qquad \max_{k=\exp} c_{ikt} \leqq \min_{k=imp,0} (-c_{ikt}) \qquad t=1,\dots,T.$$

On the macroeconomic scale that central program is regarded as optimal under which the sum of the maximal value of the sectoral objective functions is maximal.

In the dual of the i-th primal sector problem according to /2.4/-/2.8/ and under the central program /directives/ denoted by $(r_{it}, z_{ijt}, w_{it})$ let ϱ_{it} be the shadow price of the central directive r_{it}, ζ_{ijt} of z_{ijt}, ω_{it} of w_{it}, while π_{it} and σ_{is} will be the shadow prices of the special conditions bounded by R_{it} and b^o_{is}. Then with the sector shadow price system $(\varrho_{it}, \zeta_{ijt}, \omega_{it}, \pi_{it}, \sigma_{is})$ as dual program, the i-th dual sector problem has the following form:

$$/2.10/ \quad \begin{cases} f_{ikt}(\pi_{it}-\rho_{it}) + \sum_{\substack{j=1\\ j\neq i}}^{n} g_{ijkt}\zeta_{ijt} + h_{ikt}\omega_{it} + \sum_{s=\mathrm{spec}} a^{o}_{iskt}\sigma_{is} \geqq c_{ikt} \\ k = \mathrm{repr}, \mathrm{exp}, \mathrm{imp}, 0; \qquad t = 1,\dots,T \end{cases}$$

$$/2.11/ \quad \begin{cases} f_{ikt}(\pi_{it}-\rho_{it}) + \sum_{j=1}^{n} g_{ijkt}\zeta_{ijt} + h_{ikt}\omega_{it} + \sum_{s=\mathrm{spec}} a^{o}_{isk}\sigma_{is} \geqq c_{ik} \\ k = \mathrm{inv}\ ; \qquad t = 1,\dots,T \end{cases}$$

$$/2.12/ \quad \begin{cases} \rho_{it} \geqq 0, \quad \zeta_{ijt} \geqq 0, \quad \omega_{it} \geqq 0, \quad \pi_{it} \geqq 0, \quad \sigma_{is} \geqq 0 \\ j = 1,\dots,n, \quad j \neq i\ ; \quad t = 1,\dots,T; \quad s = \mathrm{spec} \end{cases}$$

$$/2.13/ \quad \sum_{t=1}^{T}\Big(R_{it}\pi_{it} - r_{it}\rho_{it} + \sum_{\substack{j=1\\ j\neq i}}^{n} z_{ijt}\zeta_{ijt} + w_{it}\omega_{it}\Big) + \sum_{s=\mathrm{spec}} b^{o}_{is}\sigma_{is} \rightarrow \min!$$

It will be left to the reader to check that the OCI problem corresponding to this two-level problem is solvable, moreover, that the polyhedral game derived from it is a regular one. /In the latter case the criterion /1.14/ and the boundedness of the set of feasible central programs may be used./ In the following the iterative procedure of the δ-termination of the regular fictitious play, i.e. of the construction of a δ-optimal macroeconomic program will be outlined /where δ is an arbitrary small positive number/.

2.2. The process of iteration

Since the set of feasible central programs is here bounded, any central program $(r_{it}^{(1)}, z_{ijt}^{(1)}, w_{it}^{(1)})$ may be made the starting point[28]. Passing over the first phases, we come to the N-th phase. From the previous $(N-1)$-th phase, 1/ the "central memory" stores the upper optimum $\phi^*\langle N-1\rangle$, the central program $(r_{it}^*\langle N-1\rangle, z_{ijt}^*\langle N-1\rangle, w_{it}^*\langle N-1\rangle)$ sent down to the sectors, the sector shadow prices $\rho_{it}^*\langle N-1\rangle, \zeta_{ijt}^*\langle N-1\rangle, \omega_{it}^*\langle N-1\rangle$ sent up from the sectors, the sector optima $\varphi_i\langle N-1\rangle$, and the mixed special sector optimum components $\phi_i^o\langle N-1\rangle$; 2/ all the "sector memories" store the shadow prices $\rho_{it}^*\langle N-1\rangle, \zeta_{ijt}^*\langle N-1\rangle, \omega_{it}^*\langle N-1\rangle$ sent up to the centre, moreover the simplex tableau and optimal base used in computing the provisional sector shadow prices $\rho_{it}^{(N-1)}, \zeta_{ijt}^{(N-1)}, \omega_{it}^{(N-1)}$. /The terms and symbols enumerated, will be defined below, in the course of the analogue terms and symbols of the N-th phase./

Step I. /Examination of the terminability of the iteration and the valuation of the sector shadow prices that are sent up, at the centre/. The formula $\varphi^*\langle N-1\rangle = \sum_{i=1}^{n} \varphi_i^{(N-1)}$ is used to calculate the lower optimum of the previous phase and this is compared with the upper optimum $\phi^*\langle N-1\rangle$, stored in the memory. Case 1. $\phi^*\langle N-1\rangle - \varphi^*\langle N-1\rangle \leqq \delta$: $N_\delta = N-1$. The iteration is terminated. The sectors are instructed to compute

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28 For in the case of a bounded $\mathcal{U}$, then $\mathcal{U}^\Delta = \mathcal{U}$. See: Goldman [7], p. 49. Corollary 1B.

the primal optimal sector programs corresponding to the provisional sector shadow prices $\varrho_{it}^{(N-1)}$, $\zeta_{ijt}^{(N-1)}$, $\omega_{it}^{(N-1)}$, using the simplex tableau and optimal base store in their memories[29]. These sector programs constitute a δ-optimal program of the long-term macroeconomic planning problem. Case 2. $\phi^*\langle N-1\rangle - \varphi^*\langle N-1\rangle > \delta$: $N_\delta > N-1$. The central valuation of the sector shadow prices that are sent up is performed, i.e. the central objective function

$$\sum_{i=1}^{n}\sum_{t=1}^{T}\Big(\varrho_{it}^*\langle N-1\rangle r_{it} + \sum_{\substack{j=1\\ j\neq i}}^{n}\zeta_{ijt}^*\langle N-1\rangle z_{ijt} + \omega_{it}^*\langle N-1\rangle w_{it}\Big)$$

is maximized underlying the conditions /2.1/-/2.3/. This linear programming problem decomposes into two types of simply solvable "micro-programming" problems: the programming

$$/2.14/\quad \begin{cases} \sum\limits_{\substack{j=1\\ j\neq i}}^{n} z_{jit} + Q_{it} = r_{it} \leqq R_{it} \\ z_{jit} \geqq 0\,,\quad j=1,\dots,n\,,\quad j\neq i\,;\quad r_{it}\geqq 0 \\ \sum\limits_{\substack{j=1\\ j\neq i}}^{n} \zeta_{jit}^*\langle N-1\rangle z_{jit} - \varrho_{it}^*\langle N-1\rangle \delta_{it} \longrightarrow \max! \end{cases}$$

which formulates the provisional production and distribution of the i-th product in the t-th period ($i=1,\dots,n$; $t=1,\dots,T$), moreover the programming

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29 The components of the optimal program may be directly read from the values of the " z -column" figuring in the column, corresponding to the slack variables of the simplex tableau. See, for instance, Karlin [9] , pp. 169-170.

$$/2.15/ \quad \begin{cases} \sum_{i=1}^{n} w_{it} = W_t \\ w_{it} \geq 0 \qquad i = 1,\dots,n \\ \sum_{i=1}^{n} \omega_{it}^{*}\langle N-1\rangle\, w_{it} \rightarrow \max! \end{cases}$$

which formulates the <u>provisional distribution of the economy's manpower quota</u> $(t = 1,\dots,T)$.

A solution of /2.14/ is as follows:

a/ if $\max_{j \neq i} \zeta_{jit}^{*}\langle N-1\rangle < \varrho_{it}^{*}\langle N-1\rangle$, then

$$/2.16/ \qquad r_{it}^{(N)} = Q_{it}\,, \quad z_{jit}^{(N)} = 0 \qquad j = 1,\dots,n\,, \quad j \neq i$$

b/ if $\max_{j \neq i} \zeta_{jit}^{*}\langle N-1\rangle = \zeta_{j_0 it}^{*}\langle N-1\rangle \geqq \varrho_{it}^{*}\langle N-1\rangle$, then[30]

$$/2.17/ \qquad r_{it}^{(N)} = R_{it}\,, \quad z_{jit}^{(N)} = \begin{cases} R_{it} - Q_{it}\,, & \text{if} \quad j = j_0 \\ 0\,, & \text{if} \quad j \neq j_0 \end{cases}$$

The solution of /2.15/ is still more simple: If

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30 If the shadow price $\zeta_{jit}^{*}\langle N-1\rangle$ is equally maximal in several sectors j /with i and t fixed/, and if it is not less than $\varrho_{it}^{*}\langle N-1\rangle$, then $R_{it} - Q_{it}$ may be partitioned between them in any proportion.

$\max_i \omega^*_{it}\langle N-1\rangle = \omega^*_{i_0 t}\langle N-1\rangle$, then[31]

/2.18/ $$w^{(N)}_{it} = \begin{cases} W_t , & \text{if} \quad i = i_0 \\ 0 , & \text{if} \quad i \neq i_0 . \end{cases}$$

It is not difficult to see that the sum of the maxima derived in the course of the micro-programs /2.14/-/2.15/ is[32]

/2.19/ $$\sum_{t=1}^{T} \left\{ \sum_{i=1}^{n} \left[\left(\max_{j\neq i} \zeta^*_{jit}\langle N-1\rangle - \varrho^*_{it}\langle N-1\rangle \right)^+ \cdot \left(R_{it} - Q_{it} \right) - - \varrho^*_{it}\langle N-1\rangle \cdot Q_{it} + \max_i \omega^*_{it}\langle N-1\rangle \cdot W_t \right] \right\} = = \phi^{\#}\langle N\rangle \quad /\text{say}/ .$$

Hence the upper optimum in the N -th phase may be computed from the formula

/2.20/ $$\phi^*\langle N\rangle = \phi^{\#}\langle N\rangle + \sum_{i=1}^{n} \phi^{o}_{i}\langle N-1\rangle .$$

Next a comparison is made of the values $\phi^*\langle N\rangle$ and $\varphi^*\langle N-1\rangle$. <u>Case 2/1</u>. $\phi^*\langle N\rangle - \varphi^*\langle N-1\rangle \leqq \delta : N_\delta = N-1$. The iteration is terminated, and the δ -optimal macroeconomic program is computed in a fashion identical with Case 1. <u>Case 2/2</u>.

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31 If the shadow price $\omega^*_{it}\langle N-1\rangle$ is equally maximal in several sectors i /with t fixed/, then W_t may be partitioned between them in any proportion.

32 α^+ is used to denote the positive part of α : $\alpha^+ = \max(\alpha, 0)$.

$\phi^*\langle N\rangle - \rho^*\langle N-1\rangle > \delta : N_\delta > N-1$. There then follows

Step II /making a new central program to be sent down to the sectors/:

$$/2.21/ \left\{ \begin{aligned} r^*_{it}\langle N\rangle &= \frac{N-1}{N} r^*_{it}\langle N-1\rangle + \frac{1}{N} r^{(N)}_{it} \\ z^*_{ijt}\langle N\rangle &= \frac{N-1}{N} z^*_{ijt}\langle N-1\rangle + \frac{1}{N} z^{(N)}_{ijt} \qquad j=1,\dots,n;\, j\neq i \\ w^*_{it}\langle N\rangle &= \frac{N-1}{N} w^*_{it}\langle N-1\rangle + \frac{1}{N} w^{(N)}_{it} \end{aligned} \right\} t=1,\dots,T.$$

Step III. /The evaluation in the sectors, of the central program component that has newly been sent down./ The linear programming problems /2.10/-/2.13/ are solved in the i-th sector, with the new objective function coefficients $r^*_{it}\langle N\rangle$, $z^*_{ijt}\langle N\rangle$, $w^*_{it}\langle N\rangle$. The simplex tableau and optimal basis[33] stored in the memory may be used for the calculation. Let the solutions

$\rho^{(N)}_{it}$, $\zeta^{(N)}_{ijt}$, $\omega^{(N)}_{it}$, $\pi^{(N)}_{it}$, $\sigma^{(N)}_{is}$ thus obtained be called provisional sector shadow prices, the minimal value of the corresponding objective function according to /2.13/ be the sector optimum, and the portion

$$/2.22/ \qquad \phi^{o(N)}_i = \sum_{t=1}^{T} R_{it}\pi^{(N)}_{it} + \sum_{s=spec} b^o_{is}\sigma^{(N)}_{is}$$

of the optimum concerning to the special shadow prices be the special sector optimum component.

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33 See, for instance, Suzuki [23] , pp. 95-96.

Step IV. /In preparation, making the new sector shadow prices and mixed special sector optimum components to be sent up to the centre in the sectors./ In the i -th sector it is necessary to compute

$$
/2.23/ \quad \left.\begin{cases}
\rho^{*}_{it}\langle N\rangle = \dfrac{N-1}{N}\,\rho^{*}_{it}\langle N-1\rangle + \dfrac{1}{N}\,\rho^{(N)}_{it} & \\[2ex]
\zeta^{*}_{ijt}\langle N\rangle = \dfrac{N-1}{N}\,\zeta^{*}_{ijt}\langle N-1\rangle + \dfrac{1}{N}\,\zeta^{(N)}_{ijt} & j=1,\dots,n,\ j\neq i \\[2ex]
\omega^{*}_{it}\langle N\rangle = \dfrac{N-1}{N}\,\omega^{*}_{it}\langle N-1\rangle + \dfrac{1}{N}\,\omega^{(N)}_{it} &
\end{cases}\right\} t=1,\dots,T,
$$

moreover the mixed special sector optimum component

$$
/2.24/ \quad \phi^{o}_{i}\langle N\rangle = \frac{N-1}{N}\,\phi^{o}_{i}\langle N-1\rangle + \frac{1}{N}\,\phi^{o(N)}_{i}\,, \qquad \phi^{o}_{i}\langle 1\rangle = \phi^{o(1)}_{i}\,.
$$

When these are sent up, the N -th iterative phase is completed.

2.3. On the economic interpretation of the model

The concrete model will now be discussed from the economic point of view. On the one hand an attempt will be made to interpret some features and properties of the model in terms of economics, and on the other some problems raised with respect to ascertaining the parameters figuring in the model.

1. What does the objective function of the dual problem of the sector models express in terms of economics? Let us presume for a moment that the centre really lets the sector have its resources at a "price" corresponding to the shadow price which the sector

reports back, and that at the same time it demands of the sector that it should not operate at a loss. If the sector reported back too high, "rosy" shadow prices, /e.g. if it stated that a rise in the manpower quota would secure greater surplus returns than it is actually capable of, according to the optimal program/, then the sector would operate at a loss. The minimization of the evaluation of the boundary conditions in terms of shadow prices, as an optimization requirement of the model, expresses the fact that care must be taken to avoid over-estimating modifications in the central directives which appear as limitations in the sector conditions, and the effect manifested in the objective function due to modification. The minimization of the dual objective function expresses an approach of careful, responsible moderation in determining the indices of economic efficiency presented as part of the report back.

2. It is a noteworthy fact that in the case of a macroeconomic planning model the game-theoretical model may be realistically interpreted and invested with economic significance. The fact is that the situation originally already shows some analogy with strategic games. Both players are in possession of certain information, but they cannot alone take fully satisfactory decisions, because to do so they would require to know the information of the other player as well. The centre has a broad purview, but it has no detailed knowledge of the special problems /e.g. the technical and cost figures for the various sectors, the special conditions limiting choice within the sector, etc./, which are known to the sectors. Or put the other way round, the sectors see many details, but they have no ability to survey the great inter-relations that can only be clear to the centre. Just as in strategic games, the situation which evolves depends on both players. Both the centre and the

sectors clearly know that the measures of the other "player" also exercise a great influence on the situation. Under such circumstances both players seek the relatively most reassuring strategy for themselves. This strategy is the "minimax" solution of the game.

In the present model the acceptance of the minimax strategy means the following: Let us presume that the centre is "omniscient" and is in possession even of those special detailed items of information that are usually only known accurately to the sectors. In this case /if ideal computing facilities were available/, it would itself be able centrally to elaborate the optimal program for the national economy /the optimal OCI program/. The program thus determined, would have a certain objective function value and result in optimal economic returns /the OCI optimum/.

If, on the other hand, the centre /both in this model and in real life/, is due to deficient information unable itself, without the collaboration of the sectors, to determine the optimal program for the economy, then the returns of the economy will be less than the optimal value. Thus in consequence of decisions taken "independently" of the sectors, relative losses will be incurred. The centre strive to cut the losses.

At the opposite end, the sectors are unable themselves, without the directing and coordinating activity of the centre, to achieve the optimal program for the economy. In the case of decisions taken "independently" of the centre, they will necessarily furnish a faulty evaluation of the resources and quotas allocated to them. Let us again presume for a moment /as was done earlier in defining the dual objective function/, that the sectors are made to a "penalty" pay for the surplus allocated to them due to the over-estimation of resources and quotas. Under such circumstances, biased evaluation, i.e. biased shadow price system, would result in a grave loss to the sector. The sectors would then obviously strive to make this loss as small as possible.

It may thus be seen that both sides strive to reduce a specific kind of relative loss. The centre's aim is that as little as possible should be lost of the optimal returns of the economy, the sector's that the optimal evaluations should be surpassed by as little as possible. The minimax solution is achieved when both players succeed in eliminating this relative loss.

3. Corresponding to Theorem 3. of this paper, a certain levelling trend of the shadow prices appears in the concrete model. First, the "demand" shadow price (ζ_{jit}) of the same proocudt i is equalized between the various sectors, as are the "demand" and "supply" shadow price ζ_{jit} and (ϱ_{it}) of the product. Second, the shadow price (ω_{it}) of the manpower quota allocated to the various sectors is equalized.

This trend fully complies with the familiar optimum condition of "welfare-economics", according to which the utilization of the marginal returns of identical resources must be equal in the various spheres[34].

The equalizing trend is of course only valid for shadow prices related to the identical period. It will also be worth studying the ratios of the shadow prices of consecutive periods, for hence it will be possible to determine a group of "discount rates".

It will be useful to study the process of equalization, and fluctuation of the shadow price system in the course of iteration.

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34 See for instance the works of Samuelson [24] and Lerner [16] .

Such an observation can reveal much, with regard to the more certain parts of the national economic plan, those that are relatively more independent of other plan targets, and the parts that are more closely dependent on other plan targets.

4. As has been pointed out, the economic policy figures of the present model are taken from the original plan, worked out by "traditional methods". Moreover, this original plan may also be chosen as the initial program for iteration. It will appear from the first steps of the iteration, whether the plan is realistic or not. If artificial variables /fictitious unbounded imports/ appear in the sector programs, the original plan was not balanced, the further steps of "two-level planning" will serve to balance it. If, however, it is not possible on the course of subsequent steps in two-level planning to eliminate the fictitious variables, then this is a warning that there is a contradiction in the economic policy figures.

"Two-level planning" thus offers an opportunity to carry out a critical check of the original plan, to discover and obviate any contradictions it may contain. As the equalization of the shadow prices of the central directives is approached /e.g. an approximately accurate knowledge of the OCI shadow prices of the external consumption Q_{it} is obtained/, so will further information become available for the critical evaluation of the economic policy figures adapted from the original plan. /E.g. to decide whether it would not be opportune to set out from a different pattern of external consumption./

5. One of the most problematic features of this concrete model is the economic content of the objective function. The optimization of the foreign trade balance as an optimum criterion, is in

the present description intended as nothing more than an example. In the course of the discussions of this problem in Hungary, other ideas have also been noted, e.g. the minimization of the total manpower expenditure, or the maximization of external consumption according to a given pattern. /This is the type of objective function recommended by Kantorowich in his work [10] ./ In both cases the economic policy targets relating to the trade balance must be incorporated in the constraint system.

It is not intended in this paper to take a definite stand over this problem - this requires many sided theoretical and practical investigation. At any rate, in the case of the first experimental computing projects it will be advisable to use several kinds of objective function and to analyse the results together.

6. Finally there is another grave problem which again can be no more than mentioned - the expression of society's time preference in the model. This is partly circumvented by prescribing external consumption separately for each period /naturally seeing that it should increase for each consecutive period, and that its pattern should change in the required manner/. It is not, however, indifferent, when the surplus returns obtained as a result of the programming will arise - whether this is to be earlier or later. It may therefore be advantageous not simply to maximize the sum of all the returns for the whole plan period, but rather some discounted total.

The other difficult question is linked to the finite duration of the plan-term. The structure of the model as described above, may involve the danger of having the program only prescribe investments whose returns appear within the plan term, and not securing a transition to the period after the plan term. This aspect would only be solved by planning for an infinite duration, a device which other considerations - particularly the difficulties of obtaining numerical data - did not yet render it desirable to adopt. For this reason, as an approximation /or we might say, by way of a compromise/, the following solution was chosen.

The requirements (Q_{it}) of external consumption are made to include the needs of the so-called "carry-over investments", i.e. those which will continue after the end of the plan term. For lack of any other source, the estimates of these figures are again adopted from the original plan. The authors are well aware of the problematic features of this solution, and the question will therefore continue to be investigated.

CONCLUSION

By way of conclusion, the following is a brief summary of the further trends of our research:

1. Our mathematical and computing research is directed mainly at elucidating how concergence in the course of "two-level planning" could be accelerated. Numerical experiments are being carried out to this end. It will require special study, whether in the case of the concrete model economic information which is otherwise available, could not be used to accelerate convergence.

2. Parallel with the numerical experiments, preparations for the practical application of the concrete model have been begun. The National Planning Bureau wishes to make use of this method too, to obtain sounder foundations for long-term macroeconomic plans. It must of course be stressed that these calculations are at present no more than experimental. They are in the stage of scientific research and can only gradually become the permanently used instruments of planning.

3. The general model described in Part 1 of the paper may also be concretely applied to other practical problems. Thus, for instance, the determination of the short-term plan of Hungarian cotton fabric exports /their composition by products and markets/, is now under preparation, using the method analogous to the two-level planning[35]. It is also intended to use the method for the elaboration of regional plans - in this case each sector corresponds to a geographic region. The authors hope that once the computing problems have been solved, the method can be widely applied.

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[35] See the paper [19] by Lipták and Nagy.

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