

description of control. The first attempts to analyze control in Leontief and von Neumann models were made by Sargan [10], Leontief [11], Lovell [12], and Jorgenson [13].

The third direction is the application of the mathematical theory of automatic control and dynamic programming to economies. Until now these have been used primarily to formalize the natural economic schemes of Keynes and the theory of growth (see, for example, Sargan [14], Lange [15], Dorfman [16], Radner [17], and Arrow and Kurz [18]). Studies of systems of economic equations based on the theory of automatic control are familiar as well (see Chou [19]). This theory has also been used in studying certain dynamic planning problems — in the work of Kendrick [20], for example. Until now, however, it has been used exclusively (for example, in [21]) to model the control mechanism for economic systems.

An attempt to unite these three directions was undertaken in a study performed by one of the authors of the present paper, J. Kornai, in cooperation with B. Martos [22, 23]; they studied autonomous control. In certain other studies — Dancs, Hunyadi, and Sivak [24], Virag [25], and Brody [26] — the Kornai-Martos model was improved. The present article belongs to this series. Its goal is to develop the investigation in the first direction involving the abstract model of the control mechanisms, by using the apparatus and results of the second (dynamic Leontief and von Neumann models) and the third (the theory of automatic control). This unification undoubtedly provides certain advantages, at the price of introducing important constraints. We have been forced to use simultaneously the strong assumptions applied both in studies of Leontief and von Neumann models and in the mathematical field of automatic controls.

2. Description of the Model (2)

General observations: Only dynamic systems are to be considered. Time is an integer variable: $t=0, 1, 2, \dots$. A unit of time is called a period. The variable representing stocks

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PROBLEMS OF CONTROL IN VON NEUMANN ECONOMIC SYSTEMS*

1. Introduction

This paper is based on the results of three directions of research. (1)

The first of these is the mathematical modeling of the control mechanism for economic systems. Experiments in this direction were begun by representatives of the neoclassical school: in the nineteenth century by Cournot [2] and Walras [3]. Then in the twentieth century, during the period of the revival of the general theory of equilibrium, such attempts were made by Samuelson [4], Arrow, Debreu [5], and their followers. In recent years, a number of authors have attempted to transfer the neoclassical schemes — in part or in whole — into the area of mathematical modeling of the control of economic systems. The most prominent among these are Hurwicz [6], as well as Marshak and Radner [7].

The second direction is dynamic Leontief and von Neumann models (see, for example, [8, 9]). In contrast to those of the aforementioned investigators, at first these models were developed only for the production sphere and did not involve a

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always applies to the start of the period (initial stocks) and flows taking place during the period.

The material sphere of the economic system is described by variables that reflect the basic material economic processes (production, production consumption, personal and social consumption, circulation) and the interrelationships between them. On the theoretical level, we also distinguish, in addition to the material sphere, the control sphere. In certain cases, membership in the material sphere is denoted by the letter $\mathcal{M}$ and in the control sphere by $\mathcal{C}$.

It is assumed that there exist n end products that are produced in n sectors.

Below we will study two types of economic systems. We will call the first a von Neumann economy since, if we consider its most important properties, it corresponds to an economy with an unchanging structure that increases at a constant rate on a von Neumann path; and the second, an ε -asymptotic von Neumann economy or, briefly, an asymptotic economy. We may also treat under the asymptotic economy the somewhat simplified dynamic Leontief system, which is rather close to the von Neumann economy with its constant structure. A more precise definition of the two economic systems will be given in more detailed discussion of the initial assumptions in the form of propositions. The modeling assumptions are almost identical in each case. In formulations of alternative assumptions and conditions that are different for the two economic systems, we use the notation N (for the von Neumann economy) and A (the asymptotic economy).

Variables. The model uses the following variables: $r(t)$ the output vector with n components (the variable r_i is the volume of output by sector i); $Y(t)$ is the matrix of purchases of dimension $n \times n$ (the variable y_{ij} is the quantity of product i , purchased by j); $w(t)$ is the vector of stocks of outputs with n components (the variable w_i represents the stock in sector i of output of the same sector); $V(t)$ is the matrix of stocks of inputs with dimension $n \times n$ (the variable v_{ij} is the stock of product i that is built up in sector j); $S(t)$ is the matrix of passive

stocks of output (3) of dimension $n \times n$ (the variable s_{ij} is equal to the excess of the total stock of output v_{ij} over the technologically necessary amount; we will return to its definition later).

The material structure. The model uses three kinds of matrices of technological coefficients. $A(t)$ is a matrix of current input coefficients of dimension $n \times n$ (it corresponds to matrix A , well known from the Leontief models). $B(t)$ is the matrix of capital input coefficients — i.e., coefficients determining the technological introduction of basic capital — of dimension $n \times n$. Actually, it corresponds to matrix B of the dynamic Leontief model, but it has an important difference: it defines only the introduction of basic capital that is logically necessary for production.

Therefore, if sector 8 produces power generators and sector 2 produces electric power, the coefficient $b_{8,2}$ describes the quantity of generators directly participating in the production of a unit of electric power without considering generator reserves. In conformity with this, $v_{ij}(t) = b_{ij}(t)r_j(t) + s_{ij}(t)$. This means that the total cost consists of two parts: technological input $b_{ij}(t)r_j(t)$ and passive stocks of outputs $s_{ij}(t)$, which serve as a reserve.

$C(t)$ is a matrix of capital input coefficients from which capital withdrawals are deducted, of dimension $n \times n$. The ratio of basic capital stock withdrawn by a particular period to the basic capital participating in production is given exogenously: $\gamma_{ij}(t)$, $0 \leq \gamma_{ij}(t) < 1$. In accordance with this $c_{ij}(t) = [1 - \gamma_{ij}(t)]b_{ij}(t)$.

The set of the three matrices of coefficients, i.e., $A(t)$, $B(t)$, $C(t)$, is called the material structure of the system.

The production balance. In order to understand better the operation of the material sphere in the model, we will write the production balance of sector i in period t in scalar notation

$$r_i(t) = \sum_{j=1}^n \underbrace{a_{ij}(t)r_j(t)}_1 + \underbrace{(w_i(t+1) - w_i(t))}_2 \quad (1)$$

$$+ \sum_{j=1}^n \underbrace{[b_{ij}(t+1)r_j(t+1) - c_{ij}(t)r_j(t)]}_3 + \sum_{j=1}^n \underbrace{[s_{ij}(t+1) - s_{ij}(t)]}_4.$$

Production is used to provide (4): 1) current production inputs; 2) increases in stocks of outputs; 3) capital investments, including expenditures for replacement of basic capital being withdrawn $[b_{ij}(t) - c_{ij}(t)]r_j(t)$ and for additional introduction of basic capital to increase production; and 4) increases in passive reserves of inputs.

The total of stocks of output and passive stocks of input is called a buffer or liability. Under ideal conditions when the economy is functioning without ups and downs, it will develop exclusively through technological inputs of products, without buffer stocks of liabilities.

Assumptions about the material sphere. R1. The closed Leontief economy. Each product is produced by one sector and each sector produces one product. Inputs are not substitutable and there is no choice among alternative technologies. There are no joint products. There are no foreign resources or final consumption. Personal consumption is considered as the input of the "labor resource sector" of the economy and labor as its output, so that these sectors are not distinguished from "the ordinary sectors" in the model. The economic system is indecomposable.

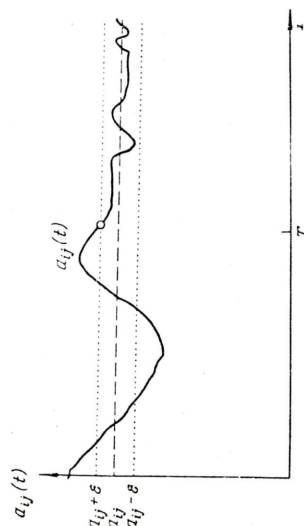
R2. Homogeneous linear input functions. Current inputs into production and technologically necessary demand for basic capital are homogeneous linear functions of output.

R3. Exogenously given coefficients of basic capital withdrawal.

R4. Productivity of the system. The system has the ability to grow. There is an excess over current inputs for construction and increases in reserves (stocks of outputs and stocks of inputs and passive stocks). In other words, the spectral radius of the matrix $A(t) + B(t) - C(t)$ is less than unity: $\sigma[A(t) + B(t) - C(t)] < 1$.

R5.N. The material structure does not change over time. The matrices of coefficients of the material structure are constant: $A(t) = A$, $B(t) = B$, $C(t) = C$.

R5. A. The material structure has an asymptotic character. The coefficient matrices of the production structure after a finite interval of time are close to some basic structure. There



Development of the "input-output" coefficients over time in the asymptotic economy.

exists a (sufficiently small) number $\epsilon > 0$ and (large) integer T for which the following relation holds: if $t > T$, then $\|A(t) - A\| < \epsilon$, $\|B(t) - B\| < \epsilon$, $\|C(t) - C\| < \epsilon$, where $\|\cdot\|$ denotes some norm of the matrix (for example, the sum of the absolute values of the elements). In the diagram the ϵ -asymptotic property is illustrated for the coefficient $a_{ij}(t)$.

Normative control. To move from the material sphere to the control sphere, we must first explain the basic idea of normative control. It is assumed that, to make decisions regarding elements of the economy (in our model, sectors), we are given certain norms that direct their behavior. In the dynamic system they are given a form of normative paths. These paths indicate the normative direction for the movement of the system.

The norms are formulated as a result of historical and social development and are fixed by agreement. In our paper the origin of the norms is not considered. This will be investigated in other studies. Here we assume that the norms are simply given.

In this paper the norms can be calculated explicitly. This is only a "sleight of hand" for the analysis, however. It does not involve any kind of institutional assumptions that someone

in fact chooses the norms of the system. In real life they are formulated experimentally as a result of impersonal social processes.

If the norms (or normative paths) are given, the control organ may indicate differences between the actual and normative values of the variables. Knowing these and counteracting the development of deviations, it prevents them and influences the system to return to the normative state. Normative control is conducted in conformity with the principle of negative feedback according to the following general control formula

$$u(t) - u^*(t) = \varphi[t, x(t) - x^*(t)], \quad \varphi(0) = 0, \quad (2)$$

$\varphi(t, \hat{x})$ decreases monotonically with respect to $\hat{x}$. Here $u(t)$ is the control of the system and $x(t)$ is the state variable. The asterisk denotes the normative value of the variable. The form of the relationship of φ must be indicated for each specific model.

In our model, the volumes of production $r(t)$ and purchases $Y(t)$ are control variables, while the stock of outputs $w(t)$ and the passive stocks of inputs $S(t)$ are state variables.

We will also introduce the abbreviated notation

$$\begin{aligned} \hat{u}(t) &= u(t) - u^*(t), \\ \hat{x}(t) &= x(t) - x^*(t). \end{aligned} \quad (3)$$

Thus, the general control formula is

$$\hat{u}(t) = \varphi[t, \hat{x}(t)]. \quad (4)$$

Assumptions concerning the control sphere. C1. Normative control. The following normative control paths are given for the system: $r^*(t)$, $Y^*(t)$, $w^*(t)$, and $S^*(t)$, which are uniformly positive variables.

C2. N. The normative path — the von Neumann path. The system has a unique von Neumann path determined in the following way:

$$\begin{aligned} \bar{r}(t) &= r_0 \lambda_0^t, & \bar{Y}(t) &= Y_0 \lambda_0^t, \\ \bar{w}(t) &= w_0 \lambda_0^t, & \bar{S}(t) &= S_0 \lambda_0^t, \end{aligned} \quad (5)$$

where λ_0 is the von Neumann growth coefficient of the system (5) $\lambda_0 > 1$.

In conformity with this assumption, the von Neumann path is normative so that

$$\begin{aligned} r^*(t) &= \bar{r}(t), & Y^*(t) &= \bar{Y}(t), \\ w^*(t) &= \bar{w}(t), & S^*(t) &= \bar{S}(t). \end{aligned} \quad (6)$$

We will denote the matrix of norms of passive stocks of inputs per unit of output by F , the matrix of norms of stocks outputs per unit of product by $\langle g \rangle$

$$S(t) = F \langle r(t) \rangle, \quad w(t) = \langle g \rangle r(t) \quad (7)$$

and the matrix of normative buffer stocks by H , i.e., $H = F + \langle g \rangle$, C2. A. The asymptotic normative path. If the initial and final production structures are given, there exists an asymptotic path denoted $\bar{r}(t)$, $Y(t)$, $\bar{w}(t)$, and $\bar{S}(t)$, on which

$$|\bar{r}(t) - \bar{r}_1(t) / \bar{r}_1(t)| < \epsilon, \quad t > T. \quad (8)$$

Similar properties are also possessed by $\bar{y}_0(t)$, $\bar{w}_1(t)$, and $s_{ij}(t)$. According to the assumptions, the asymptotic path is normative, i.e.,

$$\begin{aligned} r^*(t) &= \bar{r}(t), & Y^*(t) &= \bar{Y}(t), \\ w^*(t) &= \bar{w}(t), & S^*(t) &= \bar{S}(t). \end{aligned} \quad (9)$$

In the asymptotic economy the matrix $F(t)$ of norms of passive stocks of inputs per unit of output and the matrix $\langle g(t) \rangle$ of norms of inputs per unit output converge ϵ -asymptotically to F and $\langle g \rangle$, respectively. In C2. N and C2. A the existence of a von Neumann path (unique) is assumed. Below, when we discuss exact formulations, we will return to it in order to show the degree to which this postulate is consistent with our assumptions about the material sphere.

C3. Decentralization of decisions. Under the given norms, a decision is completely decentralized. Sector j makes a decision about the level of production of r_j and for each i about the volume of purchases y_{ij} .

C4. The absence of a memory. In our decentralized control variant, the values of the vectors are not recalled; i.e., their values derived at the preceding period are not used as information.

C5. Decentralization of information. When the norms are given, information is completely decentralized. Sector j controls decision making only for its own state variables. We will call a system of control in which C3 and C5 are simultaneously valid, autonomous (vegetative) control.

C6. Limitation of information concerning stocks. Assumption C5 means that sector j controls exclusively its own stocks of outputs w_j and the passive stocks of inputs s_i , $i=1, 2, \dots, n$.

C7. Linearity of the control law. Control is carried out according to the linear formula $\dot{u}(t) = -\alpha(t)\hat{x}(t)$, where $\alpha(t)$ is the matrix of speeds of control. Below, in formulas (12) and (13), $d(t)$ and $E(t)$ determine the speed of control.

A brief restatement of the model. The dynamic system is described by the following four equations:

a) stocks of outputs (6)

$$w(t+1) = w(t) - Y(t)e + r(t); \quad (10)$$

b) passive stocks of inputs

$$S(t+1) = S(t) - B(t+1)\langle r(t+1) \rangle - [A(t) - C(t)]\langle r(t) \rangle + Y(t); \quad (11)$$

c) control of production

$$r(t) = r^*(t) - \langle d(t) \rangle [w(t) - w^*(t)]; \quad (12)$$

d) control of purchases

$$Y(t) = Y^*(t) - E(t) \otimes [S(t) - S^*(t)]. \quad (13)$$

There are $2n+n^2$ equations in all. Equations (10) and (11) determine the laws of motion of the system, and (12) and (13) describe its control. From (10) and (11), on the basis of the control variables and the state variables for period t , we get new state variables for the period $(t+1)$. From (12) and (13), by the

deviations between the actual and normative values of the state variables, we calculate the deviations of the control variables from the normatives. The actual order of calculation will be different: (10), (12), (11), and (13). The control relates to period $(t+1)$.

3. Propositions

Below we will consider the propositions and give a short economic interpretation. The mathematical proofs are given in section 6.

Proposition 1N. Existence of a normative path. A von Neumann path exists and is unique. By solving the eigenvalue or eigenvector problem $\lambda_0(B+H)r_0 = (I-A+C+H)r_0$ and $e'r_0=1$, we get a unique vector of positive outputs $r_0 > 0$ and a growth coefficient $\lambda_0 > 1$.

If these values are known, we can calculate through matrix F and $\langle g \rangle$ the stocks of inputs $w(t)$, the passive stocks of outputs $S(t)$, and, from equation (13), the von Neumann path of purchases.

Proposition 1A. Existence of a normative path. A region of asymptotic paths exists. For each asymptotic path, the growth coefficient for any variable is contained within an ε -neighborhood of the corresponding growth coefficient for the von Neumann path that represents the basic structure.

Actually, proposition 1N establishes that, for the von Neumann economy, assumptions R1-R4, R5, and C1, as well as C3-C7, are mutually compatible. Similarly, proposition 1A may be interpreted as compatibility of the assumptions concerning the asymptotic economy. In addition, for the von Neumann economy, we have derived a method for explicit calculation of the normative path. This is a justified point for calculation of the asymptotic path.

Proposition 2N. Buffer stocks and the rate of growth. We will compare the two economic systems 1 and 2, which are completely identical except for the buffer stock norms: $h_{ij}^{(1)} \geq h_{ij}^{(2)}$ for each i , and at least one or several components in

economic system 1 are strictly greater than in economic system 2. In this case, the growth coefficient of system 1 is less than the growth coefficient of 2: $\lambda_0^{(1)} < \lambda_0^{(2)}$. (7)

This statement establishes that, in the framework of a deterministic model, an increase in the buffer stock norms reduces the rate of growth of the economy. In fact, an increase in the buffer stock norms has definite advantages: it helps to eliminate random fluctuations in production, and it ensures a smoother adaptation process. This question will be considered in section 5.

Definition. The von Neumann economy and the asymptotic economy will be called (relatively) stable if, starting with an arbitrary initial point, the variables (pairwise ratios of variables) for some period converge to a normative path (or fall within an ϵ -neighborhood of the corresponding ratios).

Proposition 3. Stability (relative). Under the control described by equations (10)-(13), the system is (relatively) stable. A sufficient condition for (relative) stability is observance of the inequalities $0 \leq d_i < 1$ and $d_i \leq \min \{(a_i + b_{ij} - c_{ij})/b_i\}$ for each i , if $b_{ij} > 0$ and $0 \leq e_{ij} \leq 1$ for each pair i, j , $1 \leq i, j \leq n$.

Proposition 3 and the definition given above with the specification in the braces are justified for the von Neumann economy and the asymptotic economy; and without the specifications in the braces, for the von Neumann economy only.

Definition. A system is called productive if not one of its variables is nonnegative and in each period at least one of the variables describing the volume of output is positive. The set of all initial points that ensure productivity of the system is called a productive initial neighborhood.

Proposition 4. Productivity. In any (relatively) stable economy we can find a productive neighborhood.

This proposition holds for the von Neumann economy and the asymptotic economy. For the first, there is a simple constructive method for calculating the productive initial neighborhood.

Propositions 3 and 4 are interpreted in the following way: for certain conditions the control described in our paper is productive and ensures relative stability.

4. The Economic Interpretation of the Propositions

Elementary quantitative adaptation. According to earlier presentations [22-25], the model described in our paper cannot be considered as a general model for control of the economy. Control of economic systems is a complex problem that can be resolved by combining different basic and auxiliary control mechanisms. Individual mechanisms control simple, short-term quantitative processes of adaptation with a very simple structure of decentralized decision making and information. In [22, 23] this type of control is called "autonomous" or "vegetative." In this paper we try to make a certain contribution to the analysis of only such "elementary" ("simple," "autonomous," "vegetative") mechanisms.

At the same time, mechanisms intended for solution of more complicated control problems also operate in any economy. Such problems include, for example, qualitative development of production and consumption, assimilation of new products; the combination of different resources and, in this connection, the selection of alternative technologies; decision making about accumulation norms; long-term adaptation and changes; and so on. Obviously, the mechanism in our study cannot be used to perform more complex control functions.

Control of the von Neumann economy. The foregoing propositions are based in part on assumptions under which the real structure of the economy does not change and the normative path is a von Neumann path. These assumptions are too strong and rather far from reality. The authors do not deny that they were forced to make these assumptions given the present state of research. Actually, since only the usual unrealistic assumptions of the Leontief model were assumed, the logic of model building almost inevitably leads to the von Neumann economy. To show the applicability of the proposed approach, we will discuss the following considerations.

The von Neumann path is a very bad model of an economic system if it is used to plan the material sphere. Actually, making the assumptions given above, we ignore the main part

of the planning problem: choosing the relative proportions of the sectors and the technologies. But the model is not so "bad" for our purposes — for theoretical analysis of control mechanisms or, more precisely, the mechanism of control that performs elementary primitive quantitative adaptations. Not only in our model, but in reality as well, solutions that can be applied for quantitative description of the usual processes of adaptation are simple, approximate methods which do not consider structural changes in the internal economic environment and react only to directly observable signals.

The process of elementary quantitative adaptation in the von Neumann economy described in our study proceeds with the utilization of stock norms similar to those observed in the real economy. The norms are fixed in the form of ratios of stocks of outputs to the volume of output or stocks of inputs to the volume of purchases ("for as many months as the stock ensures production..."). Until now, study of norms in such simple forms has been successful only for the von Neumann economy.

This elementary control mechanism may lead to indirect decisions if there are major qualitative changes, changes in the proportions of the structure of final resources and final consumption, as well as changes in technology. In such cases, the boundaries of elementary control described in this article are reached. We must introduce certain higher-order regulators, on the one hand, and the norms probably should change, on the other. We believe, however, that this is not a defect in the model. We are attempting to represent not a universal mechanism, uniquely suitable for the solution of any control problem, but a special mechanism through which certain elementary problems can be solved. Naturally, the control parameters that figure in this mechanism, the norms, and the speed of regulation must be reviewed periodically.

Control of the asymptotic economy. Assumptions concerning the asymptotic economy are weaker than those considered in the preceding section. The asymptotic economy is also an abstraction, although it is closer to reality. Technological change, shifts in proportions, which (if we consider the long-

term tendency) are characterized by a certain regularity, may be described with a rather good degree of accuracy as an approximation to some basic structure.

5. Similar Models

In this part of the paper we consider briefly two models close to the one proposed: a model of control by means of orders for stocks, and stochastic disturbances in the controls.

The model with orders for stocks. In the previously described model, commercial transactions are regulated by the buyers; in this one, they are regulated by the sellers. The choice of $y_{ij}(t)$ was performed by sector j earlier, but by sector i here. The buyers influence commercial transactions only through orders; $z_i(t)$ is the order of sector i to sector i . It is assumed that the economy operates with zero stocks of outputs. Therefore, instead of these we consider variables opposite in value — to wit, volumes of unfilled orders (we are not assuming that the orders are eliminated, although we are allowing negative orders in the form of cancellations). Negative stocks due to unfilled orders may be considered as preliminary shipments of products. The remaining part of the model is identical to the corresponding part of the model with stocks.

A short description of the model:

a) zero stock of outputs

$$r(t) = Y(t)e; \quad (14)$$

b) the volume of unfilled orders

$$K(t+1) = K(t) - Y(t) + Z(t); \quad (15)$$

c) passive stocks of inputs

$$S(t+1) = S(t) + [A(t) - C(t)] \langle r(t) \rangle - B(t+1) \times \langle r(t+1) \rangle + Y(t); \quad (16)$$

d) control of production and shipments

$$Y(t) = Y^*(t) + D(t) \otimes [K(t) - K^*(t)]; \quad (17)$$

e) control of orders

$$Z(t) = Z^*(t) - E(t) \otimes [S(t) - S^*(t)]. \quad (18)$$

The normative paths are given: $Y^*(t)$, $Z^*(t)$, $K^*(t)$, and $S^*(t)$. For the von Neumann path, $K(t) = G\bar{r}(t)$, and so forth.

The propositions given in section 3 are almost unchanged in this case. Below we will consider only those modifications that relate to the von Neumann economy.

Proposition 1N'. $H = F$. The matrix of shipments does not depend on the matrices of orders and volumes of unfilled orders. The matrix of newly arising deficits is defined as $Y_0 - Z_0 = (\lambda_0 - 1)G \otimes Y_0$.

Proposition 2N'. We will compare economic systems 1 and 2, which are completely identical except for the output stock norms: in the first, the norms are positive, and in the second, they are equal to zero. In this case $\lambda_0^{(1)} < \lambda_0^{(2)}$. From this follows the main advantage of the positive output stock norms: any kind of deficit is precluded on the von Neumann path. In addition, the greater the volume of unfilled orders, the greater the newly arising deficit (as well as the total deficit).

Proposition 3N'—4N'. There exists a similar, although not identical, sufficient condition imposed upon D and E which guarantees stability. Other details are omitted.

Stochastic disturbances in the control. As has already been noted, an increase in the buffer stock norms promotes elimination of the random disturbances in production. To analyze this problem, we introduce the following modifications: let $[\rho_i(t)]$ be an n -dimensional vector and $[\eta_i(t)]$, a matrix of dimension $n \times n$. Each variable is a random variable uniformly distributed over the entire interval $[1 - \alpha, 1 + \alpha]$, $t = 1, 2, \dots$. We will assume that the numbers of this sequence are independent. The only modification is the elementwise logical product of the right-hand side of (12) and (13) on $\langle \rho_i(t) \rangle$ and $[\eta_i(t)]$, respectively.

The new stochastic system turns out to be too complicated for theoretical analysis. To evaluate the behavior of the system, J. Kapitana used the Monte Carlo method (experiments were performed on a three-sector model based on modified data for Hungary). The following basic qualitative results were reached: 1) for the usual interval of disturbances

($\alpha \sim 0.05 \sim \lambda_0 - 1$), the random paths were linear over the entire interval with a sufficient degree of accuracy; 2) the higher the buffer stock norms, the more probable it was that the model was productive.

6. Mathematical Proofs

Proof of proposition 1N. If the von Neumann path exists, putting the relationships from C2. N into the balance of production and dividing it by λ_0 , we get the following equation:

$$r_0 = Ar_0 + (\lambda_0 - 1)\langle g \rangle r_0 + (\lambda_0 B - C)r_0 + (\lambda_0 - 1)Fr_0. \quad (19)$$

After transformation, we come to the problem of determining the eigenvalue and the eigenvector formulated in proposition 1N.

Thus, we must prove that one and only one vector $r_0 > 0$ with norm $e'r_0 = 1$ satisfies this problem.

For proof, we appeal to the fixed point problem given by (19). Since $\lambda_0 \geq 1$ and $B \geq C$, the matrix in the right-hand side is positive and all its elements increase with an increase in λ_0 . From the theory of positive matrices (see, for example, Morishima [27], Brody [28]), we have the following lemma.

Lemma 1. If U_1, U_2 are indecomposable matrices with non-negative elements $n \times n$ and $U_1 > U_2$, then $\sigma(U_1) > \sigma(U_2)$.

In this case the spectral radius of the corresponding matrix will also increase continuously. When $\lambda_0 = 1$, the spectral radius is equal to $\sigma(A + B - C)$, and by assumption $R\lambda$ is less than 1. On the other hand, the matrix being studied is greater than $(\lambda_0 - 1)U_1$, i.e., its spectral radius is also greater than $(\lambda_0 - 1)\sigma(H)$. Since the limit of the latter variable at $+\infty$ is equal to $+\infty$, the same is true for the limit of the first variables. By the theory of Bolzano there exists a unique value of λ_0 that occurs in the interval $(1, +\infty)$ and satisfies our problem together with the positive vector r_0 , which is also determined uniquely, since $e'r = 1$. In accordance with C2. N, r_0 uniquely determines w_0 and S_0 . Since (11) holds, we also uniquely determine Y_0 : $Y_0 = (\lambda_0 - 1)S_0 + (A + \lambda_0 B - C)\langle r_0 \rangle > 0$.

Proof of proposition 1A. We return to the use of stocks of

paths, it follows that $(1-\varepsilon)\bar{z}_n(t) < \bar{z}_n(t) < (1+\varepsilon)\bar{z}_n(t)$ and $(1-\varepsilon)\bar{z}_n(t)\lambda_0 < \bar{z}_n(t+1) < (1+\varepsilon)\bar{z}_n(t)\lambda_0$.

By comparing the two pairs of inequalities, we get directly

$$(1-2\varepsilon)\lambda_0 < \frac{\bar{z}_n(t+1)}{\bar{z}_n(t)} < (1+2\varepsilon)\lambda_0,$$

whence follows the proof of the proposition.

Proof of proposition 2N. In the fixed point problem (19), $H = \langle g \rangle + F$. If $H^{(1)} \geq H^{(2)}$, by lemma 1 for every $\lambda_0 > 1$ we have

$$\sigma(A + \lambda_0 B - C + (\lambda_0 - 1)H^{(1)}) > \sigma(A + \lambda_0 B - C + (\lambda_0 - 1)H^{(2)}).$$

Therefore, $\sigma\{A + \lambda_0 B - C + (\lambda_0^{(1)} - 1)H^{(2)}\} < 1$, when $\lim_{\lambda \rightarrow \infty} \sigma\{A + \lambda B - C + (\lambda - 1)H^{(2)}\} = +\infty$, i.e., by the theorem of Bolzano, $\lambda_0^{(2)} > \lambda_0^{(1)}$.

Proof of proposition 3N and 3A. The following monotonicity property plays an important role in the study of relative stability and productivity.

Lemma 2. If $0 \leq d_j \leq \min\{(a_{ij} + b_{ij} - c_{ij})/b_{ij} \text{ for all } i, \text{ for which } b_{ij} > 0\}$,

$$d_j < 1, j=1, 2, \dots, n, \text{ and } 0 \leq e_{ij} \leq 1$$

for each pair (i, j) , then from the relations

$$w^{(1)}(0) \geq w^{(2)}(0) \text{ and } S^{(1)}(0) \geq S^{(2)}(0)$$

the inequalities $w^{(1)}(t) \geq w^{(2)}(t)$ and $S^{(1)}(t) \geq S^{(2)}(t)$ follow for $t=1, 2, \dots$. In this case, the structures and rate of control of systems 1 and 2 are identical, but the initial state of system 1 is preferable.

Proof of lemma 2. We will introduce the von Neumann path into relation (10) and deduct the resulting expression from the initial one

$$\hat{w}_j(t+1) = \hat{w}_j(t) - \sum_{k=1}^n \hat{y}_{jk}(t) + \hat{r}_j(t).$$

Putting (12) into the new equation

inputs according to the relationship $V(t) = S(t) + B(t)\langle r(t) \rangle$. Actually, we can assert more than was stated earlier.

For an arbitrary material structure, assumptions $R1-R4$ guarantee the existence of a sequence $\{r^*(t), Y^*(t)\}_{t=0}^{\infty}$, that depends uniquely on the initial state $[w^*(0), V^*(0)]$, on the assumption that these variables grow at an identical rate

$$w^*(t+1) = \lambda(t)w^*(t) \text{ and } V^*(t+1) = \lambda(t)V^*(t), \\ 1 < \lambda(t) < \bar{\lambda}(t),$$

where $\bar{\lambda}(t)$ will be defined below.

To prove this proposition, we will write down the productive balance once again, but instead of $S(t)$ we will use $V(t)$: $r(t) = A(t)r(t) + [B(t) - C(t)]r(t) + w(t+1) - w(t) + V(t+1) - V(t)$, where the second term represents the deterioration. According to the proportionality assumption,

$$[I - A(t) - B(t) + C(t)]r(t) = [w^*(t) + V^*(t)e][\lambda(t) - 1]. \quad (20)$$

This equation, by assumption $R4$, has a unique solution that incidentally is positive. It is denoted by $r^*(t)$. Let

$$Y^*(t) = [A(t) + B(t) - C(t)]r^*(t) + [\lambda(t) - 1]V^*(t). \quad (21)$$

As long as (20) and (21) are equivalent to (10) and (11), then, if the stocks of produced outputs and inputs grow at the common rate $\lambda(t)$, relations (20) and (21) imply their growth at this rate.

The variables $r^*(t)$, $Y^*(t)$ are positive since $w^*(t)$ and $V^*(t)$ are positive. According to (20), inequality $S(t) \geq 0$ is equivalent to the following: $[\lambda(t) - 1]B(t) < [I - A(t) + B(t) - C(t)]^{-1} \times [w(t) + V(t)e] \leq V(t)$. The largest value of $\lambda(t)$ satisfying this inequality is denoted $\bar{\lambda}(t)$ and, in accordance with $R4$, $\bar{\lambda}(t) > 1$.

The asymptotic growth coefficient of the system will be determined only with a precision up to ε and only when the coefficients of growth for the individual variables do not differ by more than $4\varepsilon\lambda_0$ with $t > T$. Denoting the μ -th coordinate of the system $1 \leq \mu \leq 2(n^2 + n)$ by z_μ , we get

$$|\bar{z}_\mu(t+1)/\bar{z}_\mu(t) - \bar{z}_\nu(t+1)/\bar{z}_\nu(t)| < 4\varepsilon\lambda_0 \text{ for } t > T.$$

From the definitions of the asymptotic and von Neumann

$$\hat{w}_j = \frac{d_j}{\lambda - 1 + d_j} \sum_{k=1}^n \frac{e_{jk}}{\lambda - 1 + e_{jk}} (a_{jk} + \lambda \hat{b}_{jk} - c_{jk}) \hat{w}_k,$$

$$w_j > 0, j=1, 2, \dots, \lambda > 0.$$

We will consider the following problem of determining the characteristic value and vector

$$\alpha_k w_j = \frac{d_j}{\lambda - 1 + d_j} \sum_{k=1}^n \frac{e_{jk}}{\lambda - 1 + e_{jk}} (a_{jk} + \lambda b_{jk} - c_{jk}) w_k,$$

$$w_j > 0, j=1, 2, \dots, \lambda > 0.$$

We note that for $\lambda = \hat{\lambda}$ the second problem reduces to the first. Let $\mu = \max_{j,k} (1 - d_j, 1 - e_{jk})$. For $d_j < \min_k (a_{jk} + b_{jk} - c_{jk}/b_{jk})$ $a_{jk} + \mu b_{jk} -$

$c_{jk} > 0$. Therefore, if $\lambda > \mu$ each coefficient used in the problem is nonnegative, and the matrix of coefficients is indecomposable and has a spectral radius α_k .

For $\lambda = \mu + 0$ at least one coefficient is equal to $+\infty$ so that $\alpha_{k+\mu} = +\infty$, as well. For $\lambda = 1$ $\sigma_1 = \sigma(A + B - C)$ is less than unity. Since α_k is a continuous function, according to the theorem of Bolzano there exists $(\hat{\lambda}, \hat{w}, \hat{S})$ with positive elements and $\hat{\lambda} < 1$.

Since the deviations tend to zero at the rate $\hat{\lambda}$, for the relative deviations, this rate is $\hat{\lambda}/\lambda_0 < \hat{\lambda}$. Thus, from absolute stability follows relative stability, but not conversely. We note, finally, that the range of speeds of control of the system defined in assumption 3N is only a part of the range for those systems for which absolute or relative stability is guaranteed.

Proof of proposition 3A. It is well known that theorems about stability which refer to difference or differential equations with constant matrices may easily be extended to equations with asymptotic matrices. This proposition was proved by Poincaré (see Gel'fand [29]) and Perron (see Bellman [30, Chap. 2, theorem 2]). On the basis of these works, we will formulate the following lemma without proof.

Lemma 3. Let $Z(t+1) = UZ(t)$ be a stable system of first-order linear differential equations with constant coefficients,

$$\hat{w}_j(t+1) = (1 - d_j) \hat{w}_j(t) + \sum_{k=1}^n e_{jk} \hat{s}_{jk}(t). \quad (22)$$

By analogy with (22), by taking account of (13), we get

$$\hat{s}_{ij}(t+1) = (1 - e_{ij}) \hat{s}_{ij}(t) + b_{ij} d_j \sum_{k=1}^n e_{jk} \hat{s}_{jk}(t) + d_j [b_{ij} (1 - d_j) + a_{ij} - c_{ij}] \hat{w}_j(t). \quad (23)$$

From the conditions of lemma 2 it follows immediately that the matrix of coefficients of dimension $(n^2 + n)$ in (22) and (23) is nonnegative and indecomposable. Monotonicity of the "system of deviations" follows immediately from this. Since the "system of deviations" is regulated by the von Neumann path, the initial system is also monotonic. For proof of nonregularity of the matrix of coefficients, it is enough to prove that the results of a transformation of a nonnegative, nonzero vector is also a nonnegative, nonzero vector. We will assume otherwise: $\hat{w}(t) \geq 0$ and $\hat{S}(t) \geq 0$, and $\hat{w}(t+1) = 0$ and $\hat{S}(t+1) = 0$. The terms in the right-hand side of formulas (22) and (23) are also equal to zero, for example, $e_{jk} \hat{s}_{jk}(t) = 0$, $(1 - e_{jk}) \hat{s}_{jk}(t) = 0$; i.e., $\hat{s}_{jk}(t) = 0$. Similarly, $(1 - d_j) \hat{w}_j(t) = 0$. Let $d_j < 1$, $\hat{w}_j(t) = 0$. Then, $\hat{w}(t) = 0$ and $\hat{S}(t) = 0$, which contradicts the assumption. Below we will assume that all elements of matrices $A(t)$, $B(t)$, and $C(t)$ are positive.

Proof of proposition 3N. Let $(\hat{\lambda}, \hat{w}, \hat{S})$ be a normed, positive, dominant solution of the linear system (22)-(23). In this case, the matrix of coefficients is nonnegative and indecomposable. Therefore, by a theorem of Frobenius, the system has a unique positive dominant solution that can be used to formulate the simple constructive condition for stability (relative) and productivity of the system.

For proof of stability, it is necessary to show that $\hat{\lambda} < 1$. Since $(\hat{\lambda}, \hat{w}, \hat{S})$ is a unique positive solution of (22)-(23), it is enough to prove the existence of a positive solution with $\hat{\lambda} < 1$.

We will put the solution $(\hat{\lambda}, \hat{w}, \hat{S})$ into system (22)-(23). After a direct calculation we get an n -dimensional nonlinear fixed point problem

i.e., $\sigma(U) < 1$. Then there exist sufficiently small $\varepsilon > 0$ and a sufficiently large integer T for which, for an arbitrary sequence of matrices $\{U(t)\}_{t=0}^{\infty}$, the system of equations $\tilde{Z}(t) = U(t)\tilde{Z}(t)$ is also stable if $\|U(t) - U\| < \varepsilon$ for any $t > T$.

We will divide the matrix of coefficients thus derived by λ_0 ; the vector of period t , by λ_0^t ; and the vector of period $(t+1)$, by λ_0^{t+1} . This means that in place of the value indicators we use the ratios, and instead of relative stability, absolute stability.

We will apply lemma 3 to the asymptotic generalization of the system of equations (22)-(23). Then, if $t > T$,

$$\begin{aligned} \|d(t) - d\| &< \varepsilon, \quad \|E(t) - E\| < \varepsilon, \quad \|A(t) - A\| < \varepsilon, \\ \|B(t) - B\| &< \varepsilon, \quad \|C(t) - C\| < \varepsilon. \end{aligned}$$

We will take a sufficiently small ε such that the ratios for the asymptotic systems converge at a rate not less than $1 - \varepsilon$.

In this case, for example, $[w_i(t) - \tilde{w}_i(t)]\lambda_0^{(t+\tau)} \rightarrow 0$, and, in addition, $w_i(t)/\lambda_0^{(t-\varepsilon)} \rightarrow 0$. From this it follows that, if $t \rightarrow \infty$, then $[w_i(t) - \tilde{w}_i(t)]/\tilde{w}_i(t) \rightarrow 0$.

The system is quite complicated if the asymptotic normative path is unattainable. The essence of the problem remains clear if we consider an arbitrary unattainable von Neumann path.

Since $\sigma(U) < 1$, the limit of the iterative process $\hat{Z}(t+1) = U\hat{Z}(t) + \hat{V}$ is determined in the form $\hat{Z} = (I - U)^{-1}\hat{V}$, which generalizes the case of feasible von Neumann paths when $\hat{V} = 0$, $\hat{Z} = 0$.

Proof of proposition 4A. In the general case it is true that, if for a relatively stable system of difference equations the equilibrium path at each moment of time is bounded below by a positive vector that is constant over time, there exists an initial neighborhood from which we can always emerge onto a path that is positive throughout the interval.

Proof of proposition 4N. For the von Neumann economy, a constructive method is proposed for determining the initial neighborhood that is maximal in a certain sense.

We need the following variables:

$$\alpha = \min_{1 \leq j \leq n} \frac{w_j^0}{\tilde{w}_j}, \quad \beta = \min_{1 \leq j \leq n} \frac{r_j^0}{d_j w_j},$$

$$\gamma = \min_{1 \leq i, j \leq n} \frac{S_{ij}^0}{\tilde{S}_{ij}}, \quad \xi = \min_{1 \leq i, j \leq n} \frac{Y_{ij}^0}{e_{ij} \tilde{S}_{ij}},$$

$$w^{(m)} = w_0 - \alpha \tilde{w}, \quad w^{(M)} = w_0 + \beta \tilde{w},$$

$$S^{(m)} = S_0 - \gamma \tilde{S} \quad \text{and} \quad S^{(M)} = S_0 + \xi \tilde{S}.$$

Lemma 4. The initial neighborhood, defined by the inequalities

$$w^{(m)} \leq w(0) \leq w^{(M)} \quad \text{and} \quad S^{(m)} \leq S(0) \leq S^{(M)},$$

is not empty and productive. An increase in any of the variables $\alpha, \beta, \gamma, \xi$ in the expressions for $w^{(m)}, w^{(M)}, S^{(m)}$, and $S^{(M)}$ leads to a violation of productivity from the very start.

Proof. We turn to the "system of deviations." Then the initial neighborhood is defined in the following way: $-\alpha \tilde{w} \leq w(0) \leq \beta \tilde{w}$, $-\gamma \tilde{S} \leq S(0) \leq \xi \tilde{S}$.

By virtue of monotonicity, for each t we have

$$-\alpha \tilde{w} \leq \tilde{w}(t) \leq \beta \tilde{w}^t \quad \text{and} \quad -\gamma \tilde{S} \leq S(t) \leq \xi \tilde{S}^t.$$

Taking account of relative stability as $0 < \lambda < \lambda_0$, and by definition we have: $-w_0 \leq \alpha \tilde{w}$, $\beta \tilde{w} \leq \langle d_j^{-1} \rangle r_0$, $-S_0 \leq -\gamma \tilde{S}$, $\xi \tilde{S} \leq [e_{ij}^{-1}] \otimes Y_0$. From this it follows that: $-w_0 \lambda_0^t \leq \tilde{w}(t) < \langle d_j^{-1} \rangle r_0$ and $-S_0 \lambda_0^t \leq [e_{ij}^{-1}] \otimes Y_0 \lambda_0^t$, $t > 0$.

These conditions, according to (12) and (13), ensure not only positiveness of $w(t)$ and $S(t)$ but positiveness of $r(t)$ and $Y(t)$ for $t > 0$ as well.

Notes

1) The authors participated in this study as follows. In 1973, J. Kornai [1] published a large study in which some of the basic ideas of this article were put forward. The model was described by the authors jointly. A. Shimonovich proved a number of mathematical theorems and Kornai gave the economic interpretation of the assumptions and the conclusions.

2) Matrices are denoted by upper-case Latin characters; matrix elements are denoted by lower-case letters with double

subscripts; vectors by lower-case Latin characters, and their components by the same methods with subscripts. An apostrophe denotes a transpose. The diagonal matrix is denoted by the vector of elements on the main diagonal included between the $()$; the logical product of two matrices (elementwise) is denoted by $\otimes$, and vector and matrix inequalities by $\geq$ and $\leq$, where, inequality of all pairs of elements is permitted in the first case and is not in the second.

3) The term "passive" (slack) refers to that part of the stock which remains unused; it does not have any negative valuation and denotes reserves that are specially created to allow for random disturbances, excesses, not justified from the viewpoint of deterministic controls.

4) For simplicity of interpretation, we are assuming that both production and reserves are growing.

5) The definition of λ_n is given below in proposition 1.

6) In the vector and matrix relations here and below, ϵ is an n -dimensional column vector, all of whose components are equal to 1.

7) With corresponding modifications, proposition 2N may be extended to the asymptotic economy as well. However, this question is not considered in this paper.

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ON RELATIVE STABILITY AND STABILIZATION OF NONLINEAR BALANCE SYSTEMS*

Dynamic balance systems have now found a wide area of application to describe the structure of production processes and interindustry relations. The best studies are linear balance systems of the Leontief type, which enable us to derive conclusions about the qualitative behavior of interindustry systems over a long time period of operation.

Analysis of interindustry relations underlies the method that has been given the name "input-output." Its basic nature consists in determining the levels of gross output for a given external final demand within the framework of the existing technological possibilities, expressed in input coefficients. In its dynamic variant, this analysis frequently reduces to study of the behavior of a system of the form

$$x_i(t+1) = \sum_{j=1}^n a_{ij}(t)x_j(t) + c_i(t), \quad i=1, \dots, n, \quad t=0, 1, 2, \dots$$

Many of the results derived under the assumption of linearity can also be generalized to nonlinear cases. Here, the expendi-

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