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Source: *Econometrica*, Vol. 33, No. 1 (Jan., 1965), pp. 141-169
Published by: The Econometric Society
Stable URL: http://www.jstor.org/stable/1911892
Accessed: 21/06/2010 10:49

Econometrica, Vol. 33, No. 1 (January, 1965)

TWO-LEVEL PLANNING

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The planning task may originally be formulated as a single linear programming problem of the maximizing type. This *overall central information (OCI) problem* may be decomposed into subproblems that can be solved by mutually independent "sectors," coordinated by the "centre" through having the latter allocate the resources to the various sectors. The original OCI problem is then transformed into a *two-level problem*, in which the "central problem" is to evolve an allocation pattern where the sum of the maximal yields of the "sector problems" will be the greatest.

The solution of the two-level problem is achieved by setting up a game-theoretical model. The players are on the one hand the centre, on the other the team of sectors. The strategies of the centre are the feasible allocation patterns, those of the sectors are the feasible shadow price systems in the duals of the sector problems. The payoff function is the sum of the dual sector objective functions. It is shown that if certain regularity conditions are satisfied, then the value of the *polyhedral game* which has thus been defined is the maximal yield of the OCI problem. In place of a direct solution of the polyhedral game, a fictitious play of the game is undertaken.

The first part of the paper discusses a general model, within whose scope the symbols and definitions are presented and the mathematical theorems are proved. In the second part, the results of the first part are applied to a long-term macroeconomic planning model.

INTRODUCTION[1]

IN RECENT years work has begun in Hungary on the application of mathematical methods to the higher levels of planning. Experiments are proceeding in two directions. One of these has been the use of mathematical programming in several sectors of industry, to form a basis for their plans. The calculations—some of which have been completed—use economic optimum criteria to determine the most favourable program for the economic activities in the particular sector being considered:[2] production, producers' utilization of capital, exports, imports, investments, etc.

The other direction has been the use of input-output tables (static Leontief models) in national planning.[3] The National Planning Bureau now makes regular use of the input-output matrix of the economy to check the inner coordination of

[1] The authors first published the method treated in this paper in duplicated form under the aegis of the Computing Centre of the Hungarian Academy of Sciences in May, 1962 [13], and, with the addition of a revised version of the mathematical part, in October, 1962 [17].

An earlier version of this paper appeared in Hungarian in the Publications of the Mathematical Institute of the Hungarian Academy of Sciences [14]. A paper by Lipták [18] discusses a further developed version of the "general model" treated here.

[2] See [12].

[3] Detailed information on the use of input-output tables in Hungary is presented in the material of the scientific conference held in Budapest in 1961; see [6].

the annual and five-year plans. This is the first mathematical tool to have been used in Hungary for the preparation of macroeconomic plans. It is, however, as is generally known, not suitable for optimization and thus only useful in achieving the correct proportions among sectors.

A survey of the situation thus leads to the next logical step—the need to devise procedures that will permit optimization, but this time for the whole of the national economy. This need is frequently expressed by the practical planners, and the Hungarian scientific literature on the subject contains such proposals. The ideas advanced so far have, however, not been able to overcome the basic difficulties: Either one sets up a highly aggregated programming model, in which case the freedom of choice is extremely narrow, and the extent of aggregation and resulting excessive simplifications endanger the utility of the computed results. Or else, if one builds a model large enough to be free of these deficiencies, then not even high-power electronic computers will be able to cope with the numerical solution of the problem.

The present research project has been aimed at decomposing the large programming scheme. This idea has appeared several times in the literature of macroeconomic planning—it is sufficient to refer to the work of Kantorowich **[9]**, of Frisch **[5]**, and to the paper of Trzeciakowski **[24]**. There are mathematical methods for the decomposition of linear programming tasks of special kinds, e.g., those in the papers of Dantzig and Wolfe **[3, 4]**, but it has been found that the known procedures do not provide solutions to this problem. Thus if the decomposition procedure of Dantzig and Wolfe were applied to the concrete macroeconomic model in hand, the "coordinating program" would still be of such size as to be unmanageable for computing with the usual processes (e.g., the simplex method). Another approach was therefore adopted.

The planning task may originally be formulated as a single linear programming problem of the maximizing type, whose size will be too great for the given computing facilities. This we shall henceforth call the *over-all central information problem* (OCI problem). The OCI problem may be decomposed into subproblems that can be solved by mutually independent "sectors," coordinated by the "centre," which allocates among the sectors the limitations prescribed in the OCI problem (the resources, materials, and labour). The original OCI problem is then transformed into a *two-level problem*, in which the "central problem" is to evolve an allocation pattern where the sum of the maximal yields of the "sector problems" will be greatest.

The solution of the two-level problem is achieved by setting up a game-theoretical model. The players are, on the one hand, the centre, on the other, the team of sectors. The strategies of the centre are the feasible allocation patterns, those of the sectors the feasible shadow price systems in the duals of the sector problems. The payoff function is the sum of the dual sector objective functions. It is shown that if certain regularity conditions are satisfied, then the value of the *polyhedral game* which has thus been defined is the maximal yield of the OCI problem, and that with the help of its optimal strategies, the solution of the OCI problem can be obtained.

In place of a direct solution of the polyhedral game, a fictitious play of the game is undertaken. In the course of this, each sector separately evaluates the suitably chosen initial allocations of the centre (by means of dual linear programming), and reports back to the centre. The centre, applying a certain procedure, correspondingly modifies its initial allocations and sends down new directives to the sectors, which again evaluate these, report back to the centre, and so on. The iteration thus obtained permits the OCI problem to be solved with any required degree of accuracy, in the sense that a sufficient number of iterations will lead to a feasible OCI program whose yield will differ from the maximum OCI yield by as small an amount as it is wished to stipulate. Since planning according to this method takes place alternately at two levels—the centre and the sectors—organically interlinked with one another, continuously supplementing and correcting each other, the authors have called their procedure *two-level planning*.

Actually, the inspiration for the development of the allocation technique of the OCI problem and the iterative method of solution was derived from the present planning practice in a Socialist economy. The method to be described is in some degree an imitation of the usual course of planning. The National Planning Bureau, acting on the basis of the requirements of economic policies and of general information about the various sectors, works out a preliminary draft plan which contains general targets (quota figures) for the sectors. The centre makes a provisional distribution of the available resources, material, manpower, etc. among the sectors, and at the same time also allocates their output targets. The sectors then proceed, through their own detailed calculations made on the basis of their concrete conditions, to give "substance" to the quotas and to lend concrete meaning to the central targets. In so doing, they also make recommendations for changes to the Planning Bureau. This is what is in economic usage called "counter-planning." On the basis of the counter-plans the National Planning Bureau modifies its original targets and again sends them down to the sectors. The method proposed here is an attempt to aid this process of planning and counter-planning by means of objective criteria.

The procedure recommended also simulates the usual practice of planning in another respect. It repeatedly happens that the centre gives the sectors certain directives and asks them to report on the degree of economic efficiency with which the task can be carried out. The sectors express the efficiency of their activities through various "indices of economic efficiency," whose structure is prescribed by the centre. The method to be treated incorporates this reporting-back process in a unified system, where the sectors at each step report back one type of economic efficiency index—the shadow prices derived from programming—to the centre for the evaluation of the directives obtained from there.

Mention has been made of the fact that, in some sectors of industry, mathematical programming methods are also being used to elaborate plans on the sector scale. In these programming models certain directives received from the National Planning Bureau—the output targets, manpower limitations, etc.—figure as constants on the right-hand side of the constraints. These programs indeed suggest that it ought to be worth while to compare the results of the sector programs and utilize them to improve the directives and quotas derived from the national plan. The function of the proposed process is to lend an organized form to such comparisons and the macroeconomic plan corrections based upon them, in fact to link up organically the programming work done at the sector levels.

The first part of this paper discusses a general model, within whose scope the symbols and definitions may be more easily presented and the mathematical theorems more easily proved. Section 1.1 details the transformation of the OCI program into a two-level one, while Section 1.2 expounds the latter's transformation into a polyhedral game and its iterative solution.[4] These sections permit an insight into how the method may be used for the solution of general linear programming problems; some questions which arise and an example are dealt with briefly in Section 1.3. In the second part of the paper, the results of the first are applied to the concrete model mentioned in the Introduction, i.e., to the problem of long-term macroeconomic planning. Section 2.1 is a description of the model, Section 2.2 presents the process of iteration, and Section 2.3 discusses some problems of economics arising in connection with the model.

1. THE GENERAL MODEL

1.1. *The transformation of the over-all central information problem into a two-level problem*

Let

(1.1) $$Ax \leqslant b,\ x \geqslant 0,\ c'x \to \max! \quad \text{and} \quad y'A \geqslant c',\ y \geqslant 0,\ y'b \to \min!$$

be the canonical forms[5] of the primal and dual versions, respectively, in the OCI problem of the general model. The primal variable of the OCI problem (the vector x) is called the OCI program, the dual variable (the vector y) is the OCI shadow price system. Let $\mathscr{X}$ denote the set of feasible OCI programs, and $\mathscr{X}^*$ the set of optimal OCI programs. Let $\mathscr{Y}$ be the set of feasible OCI shadow price systems and

[4] The symbols used in these sections are as follows: Greek letters denote real numbers; small Latin letters (except i and n) mean vectors; i, n, and N are positive integers; capital Latin letters (except N) denote matrices. The prime is used for transposition. A vector (if not qualified) is understood to mean a column vector. Script characters in capitals are used to indicate sets.

[5] The primal-dual versions of all linear programming problems can be transformed into the symmetrical form (1.1).

$\mathscr{Y}^*$ the set of optimal OCI shadow price systems,[6] i.e.,

$$(1.2) \qquad \mathscr{X}=\{x:Ax\leqslant b,\ x\geqslant 0\}, \qquad \mathscr{X}^*=\{x^*:x^*\in\mathscr{X},\ c'x^*=\max_{x\in\mathscr{X}} c'x\}\,,$$

$$(1.3) \qquad \mathscr{Y}=\{y:y'A\geqslant c',\ y\geqslant 0\}, \qquad \mathscr{Y}^*=\{y^*:y^*\in\mathscr{Y},\ y^{*\prime}b=\min_{y\in\mathscr{Y}} y'b\}\,.$$

Let it be assumed that the OCI problem is solvable, i.e., that an optimal OCI program exists: $\mathscr{X}^*\neq\emptyset$.[7] It is known then that there also exists an optimal OCI shadow price system[8]: $\mathscr{Y}^*\neq\emptyset$. Moreover, the maximum value of the objective function in the primal version and the minimum value of the objective function in the dual version are equal—their common value is the optimum Φ of the OCI problem:

$$(1.4) \qquad \max_{x\in\mathscr{X}} c'x = \min_{y\in\mathscr{Y}} y'b=c'x^*=y^{*\prime}b=\Phi \qquad (x^*\in\mathscr{X}^*,\ y^*\in\mathscr{Y}^*)\,.$$

The solvability of the OCI problem is incidentally equivalent to the assumption that there exists a feasible OCI program and also a feasible OCI shadow price system,[9] i.e.,

$$(1.5) \qquad \mathscr{X}\neq\emptyset \quad \text{and} \quad \mathscr{Y}\neq\emptyset\,.$$

Let

$$(1.6) \qquad A=[A_1,\ldots,A_n], \qquad x=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}, \qquad c'=[c'_1,\ldots,c'_n]$$

be the mutually corresponding partitioning of the matrix A, the OCI program x, and the OCI objective function-vector c' in the primal version of the OCI problem. Then in place of (1.1) the equivalent forms

$$(1.7) \qquad \left\{\begin{array}{llll} A_1x_1+ & \ldots & +A_nx_n & \geqslant 0\\ x_1 & & & \geqslant 0\\ & \ddots & & \\ & & x_n & \geqslant 0\\ c'_1x_1+ & \ldots & +c'_nx_n & \to\max! \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{ll} y'A_1 & \geqslant c'_1\\ \ddots & \\ y'A_n & \geqslant c'_n\\ y & \geqslant 0\\ y'b & \to\min! \end{array}\right\}$$

may be used.

If the sum of the vectors $u_1,\ldots,u_n$ (of the same size as the bounding vector b) is itself b, i.e., if it satisfies the bounding vector partitioning condition

$$(1.8) \qquad u_1+\ldots+u_n=b\,,$$

[6] In the case of elements z of arbitrary nature, $\{z:\quad\}$ denotes the set of those elements z which satisfy the condition following the colon.

[7] $\emptyset$ is the symbol for the empty set.

[8] Goldman and Tucker [**8**, Corollary 1A, p. 60].

[9] Goldman and Tucker [**8**, Theorem 2, p. 61].

then the vector

$$(1.9) \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

composed of them is called a central program, the vector u_i is the ith sector component of the central program u, while the ith sector problem under the central program u (or under the sector component u_i) is understood to mean the linear programming problem

$$(1.10) \qquad A_i x_i \leqslant u_i,\ x_i \geqslant 0,\ c_i' x_i \to \max! \quad \text{and} \quad y_i' A_i \geqslant c_i',\ y \geqslant 0,\ y' b \to \min!$$

In (1.10) x_i is the ith sector program, while y_i is the ith sector shadow price system. In the ith sector problem under the sector component u_i, let $\mathscr{X}_i(u_i)$ stand for the set of feasible sector programs, $\mathscr{X}_i^*(u_i)$ for the set of optimal sector programs, $\mathscr{Y}_i$ for the set of feasible sector shadow price systems, and $\mathscr{Y}_i^*(u_i)$ for the set of optimal sector shadow price systems, i.e.:

$$(1.11) \qquad \left[\begin{array}{l} \mathscr{X}_i(u_i) = \{x_i : A_i x_i \leqslant u_i,\ x_i \geqslant 0\}, \\ \mathscr{X}_i^*(u_i) = \{x_i^* : x_i^* \in \mathscr{X}_i(u_i),\ c_i' x_i^* = \max\limits_{x_i \in \mathscr{X}_i(u_i)} c_i' x_i\}; \end{array}\right.$$

$$(1.12) \qquad \left[\begin{array}{l} \mathscr{Y}_i \quad\ \ = \{y_i : y_i' A_i \geqslant c_i',\ y_i \geqslant 0\}, \\ \mathscr{Y}_i^*(u_i) = \{y_i^* : y_i^* \in \mathscr{Y}_i,\ y_i^{*\prime} u_i = \min\limits_{y_i \in \mathscr{Y}_i} y_i' u_i\} \qquad (i = 1, \ldots, n). \end{array}\right.$$

Let us find the condition for the solvability of all the sector problems. Since it follows from (1.12) and (1.3) that $\mathscr{Y} = \mathscr{Y}_1 \cap \ldots \cap \mathscr{Y}_n$, and, because according to the assumption made with regard to the solvability of the OCI problem, (1.5) states that $\mathscr{Y} \neq \emptyset$, therefore

$$(1.13) \qquad \mathscr{Y}_i \neq \emptyset \qquad (i = 1, \ldots, n).$$

Hence two necessary and sufficient conditions may be deduced for the solvability of the ith sector programming problem under u_i. The first is that

$$(1.14) \qquad \mathscr{X}_i(u_i) \neq \emptyset\,.$$ [9]

The second is that $y_i' u_i$ is bounded from below on the set $\mathscr{Y}_i$.[10] This latter statement is best put in another form. Let

$$(1.15) \qquad \mathscr{Y}_i = Y_i^{\triangle} + \overline{Y}_i^{<} = \{Y_i q_i + \mu_i \overline{Y}_i \overline{q}_i : 1' q_i = 1' \overline{q}_i = 1,\ q_i \geqslant 0,\ \overline{q}_i \geqslant 0,\ \mu_i \geqslant 0\}$$

[10] Goldman and Tucker [8, Corollary 1B, p. 60].

be the canonical decomposition[11] of the polyhedral set $\mathscr{Y}_i$. Then the boundedness condition of $y_i'u_i$ on the set $\mathscr{Y}_i$ may be written in the form

$$(1.16) \qquad \bar{Y}_i'u_i \geqslant 0 .$$

Let those central programs for which all the sector problems are solvable be called *evaluable central programs.* According to (1.8) and (1.16) the set $\dot{\mathscr{U}}$ of the evaluable central programs can be written in the form,

$$(1.17) \qquad \dot{\mathscr{U}} = \left\{ \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} : u_1 + \ldots + u_n = b, \quad \bar{Y}_1'u_1 \geqslant 0, \ldots, \bar{Y}_n'u_n \geqslant 0 \right\} .$$

$\dot{\mathscr{U}}$ is therefore a polyhedral convex set.

Let $\mathscr{X}(u)$ denote those OCI programs which may be composed from the sector programs feasible under the evaluable central program $u = [u_1', \ldots, u_n']'$. Then

$$(1.18) \qquad \mathscr{X}(u) = \mathscr{X}_1(u_1) \times \ldots \times \mathscr{X}_n(u_n) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1 \in \mathscr{X}_1(u_1), \ldots, x_n \in \mathscr{X}_n(u_n) \right\} .$$ [12]

The (proper or improper) subset $\mathscr{U}$ of the set $\dot{\mathscr{U}}$ consisting of all the evaluable central programs, is said to *generate* $\mathscr{X}$, or in other words $\mathscr{U}$ is the *generating set of* $\mathscr{X}$ if

$$(1.19) \qquad \mathscr{X} = \bigcup_{u \in \mathscr{U}} \mathscr{X}(u) .$$

For the case of $\mathscr{U} = \dot{\mathscr{U}}$, (1.19) is valid, i.e., the non-void polyhedral convex set $\dot{\mathscr{U}}$ consisting of all the evaluable central programs generates the set $\mathscr{X}$ of the feasible OCI programs. This may be proved in two steps: (1) From (1.5), $\mathscr{X} \neq \emptyset$. Let $x = [x_1', \ldots, x_n']' \in \mathscr{X}$, and let the components $u_1 = A_1x_1, \ldots, u_{n-1} = A_{n-1}x_{n-1}$, $u_n = b - (u_1 + \ldots + u_{n-1})$ be defined. Obviously $x_i \in \mathscr{X}_i(u_i)$, so that $\mathscr{X}_i(u_i) \neq \emptyset$ $(i = 1, \ldots, n)$, and consequently, because of (1.14), $u = [u_1', \ldots, u_n']'$ is an evaluable central program, so that $\dot{\mathscr{U}} \neq \emptyset$. $\dot{\mathscr{U}}$ is therefore a non-void set. Moreover, $x \in \mathscr{X}(u)$, and since the above construction provides such a central program, $u \in \dot{\mathscr{U}}$ for each OCI program $x \in \mathscr{X}$; therefore $\mathscr{X} \subset \bigcup_{u \in \dot{\mathscr{U}}} \mathscr{X}(u)$. (2) If $u = [u_1', \ldots, u_n']' \in \dot{\mathscr{U}}$ and $x_i \in \mathscr{X}_i(u_i)$,

[11] Goldman [7, pp. 44–49.] $1'$ denotes a row vector whose every component is 1. In (1.15) therefore, q_i and $\bar{q}_i$ are socalled probability vectors, i.e., nonnegative vectors, the sum of whose components equals 1. Y_i is the matrix consisting of the extreme points of $\mathscr{Y}_i$. If $\mathscr{Y}_i$ is bounded, then by definition $\bar{Y}_i = 0$. Otherwise, $\bar{Y}_i$ is the matrix consisting of the extreme vectors of the set $\bar{\mathscr{Y}}_i$, which consists of the probability vectors satisfying the reduced homogeneous system $y_i' A_i \geqslant 0$, $y_i \geqslant 0$.

[12] $\mathscr{A} \times \mathscr{B} \times \ldots$ is the direct product of the sets $\mathscr{A}, \mathscr{B}, \ldots$. If $\mathscr{A}, \mathscr{B}, \ldots$ are sets in column vector spaces, then the general element of the direct product set is the column vector composed in the manner shown by (1.18).

i.e., if $A_i x_i \leqslant u_i$, $x_i \geqslant 0$ $(i=1, \ldots, n)$, then because of the bounding vector partitioning condition (1.8), $A_1 x_1 + \ldots + A_n x_n \leqslant u_1 + \ldots + u_n = b$, $x = [x_1', \ldots, x_n']' \geqslant 0$, so that $x \in \mathscr{X}$. Hence $\mathscr{X}(u) \subset \mathscr{X}$, and $\bigcup_{u \in \dot{\mathscr{U}}} \mathscr{X}(u) \subset \mathscr{X}$. This completes the proof.

The generating property of (1.19) may also be a property of a non-void proper polyhedral subset of the set $\dot{\mathscr{U}}$. By way of an example actually used in the model, consider the following: In the case of a matrix A of special form, (1.7) may assume the form

$$(1.20) \qquad A_1^{\#} x_1 + \ldots + A_n^{\#} x_n \leqslant b^{\#} ,$$

$$(1.21) \qquad \begin{cases} A_1^0 x_1 & & \leqslant b_1^0 \\ & \ddots & \\ & & A_n^0 x_n \leqslant b_n^0 , \end{cases}$$

$$(1.22) \qquad x_1 \geqslant 0, \ldots, x_n \geqslant 0 ,$$

$$(1.23) \qquad c_1' x_1 + \ldots + c_n' x_n \to \max! .$$

(Here $A_i^0 x_i \leqslant b_i^0$ comprises those conditions of the OCI problem which only refer to the ith sector. These conditions may be called the special sector conditions of the ith sector.) It is useful here also to put the central program in the form

$$(1.24) \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad u_i = \begin{bmatrix} u_i^{\#} \\ u_{i1}^0 \\ \vdots \\ u_{in}^0 \end{bmatrix} \qquad (i=1, \ldots, n) ,$$

and in this case the bounding vector partitioning condition is expressed by the equations

$$(1.25) \qquad u_1^{\#} + \ldots + u_n^{\#} = b^{\#} ,$$

$$(1.26) \qquad u_{1i}^0 + \ldots + u_{ni}^0 = b_i^0 \qquad (i=1, \ldots, n) .$$

Let $\dot{\mathscr{U}}$ here again stand for the set of all the evaluable central programs, and $\mathscr{U}$ here denote that subset of $\dot{\mathscr{U}}$ in which each sector attains in full the bounds occurring in its special sector conditions, that is,

$$(1.27) \qquad \mathscr{U} = \left\{ u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} : u \in \dot{\mathscr{U}}, \quad u_1 = \begin{bmatrix} u_1^{\#} \\ b_1^0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, u_n = \begin{bmatrix} u_n^{\#} \\ 0 \\ \vdots \\ 0 \\ b_n^0 \end{bmatrix} \right\} .$$

It is obvious that $\mathscr{U}$ is a non-void convex polyhedral subset of $\dot{\mathscr{U}}$, which does not necessarily comprise every evaluable central program.[13]

Returning to the general case, let $\mathscr{U}$ now be a non-void convex polyhedral subset of $\dot{\mathscr{U}}$ such that it generates $\mathscr{X}$. Let us fix $\mathscr{U}$ and let its elements be called *feasible central programs.* For any feasible central program $u=[u_1',\ldots,u_n']'$, the *sector-optima* $\varphi_i(u_i)=\max_{x_i\in\mathscr{X}_i(u_i)} c_i'x_i=\min_{y_i\in\mathscr{Y}_i} y_i'u_i$ $(i=1,\ldots,n)$, and their sum, the over-all optimum $\varphi(u)=\varphi_1(u_1)+\ldots+\varphi_n(u_n)$ under u may be defined. These are continuous and piecewise linear concave functions of u. For if $y_{i1},\ldots,y_{iN_i}$ denote the extreme points of $\mathscr{Y}_i$, i.e., if $Y_i=[y_{i1},\ldots,y_{iN_i}]$ can be substituted in (1.15), then because u is evaluable, $\varphi_i(u_i)=\min_{y_i\in\mathscr{Y}_i} y_i'u_i=\min\{y_{i1}'u_i,\ldots,y_{iN_i}'u_i\}$, so that it is the lower envelope of a finite number of linear functions $(i=1,\ldots,n)$.

The two-level problem obtained from the OCI problem by means of the sector decomposition (1.6) and choice of the set of feasible central programs as above is understood to mean a problem as follows.

(1) At the "central level," to determine the feasible central program(s) which yield(s) the maximal over-all optimum, in other words to solve the concave programming problem, $u\in\mathscr{U}$, $\varphi(u)\to\max!$ and to determine the set

$$(1.28)\qquad \mathscr{U}^*=\{u^*:\varphi(u^*)=\max_{u\in\mathscr{U}}\varphi(u)\},$$

consisting of the optimal central programs.

(2) At the "sector level," to determine in each sector the optimal sector program(s) belonging to the optimal central program component(s), i.e., for each $u^*=[u_1^{*\prime},\ldots,u_n^{*\prime}]'\in\mathscr{U}^*$ to solve the sector problems $A_ix_i\leqslant u_i^*$, $x_i\geqslant 0$, $c_i'x_i\to\max!$ —thus to determine the sets $\mathscr{X}_i^*(u_i^*)$ for $i=1,\ldots,n$, defined in (1.11).

(3) To compose the OCI program(s) which can be obtained from the optimal sector programs under the optimal central program(s); in other words, to determine the union of the sets

$$(1.29)\qquad \mathscr{X}^*(u^*)=\mathscr{X}_1^*(u_1^*)\times\ldots\times\mathscr{X}_n^*(u_n^*)=\left\{\begin{bmatrix}x_1^*\\ \vdots\\ x_n^*\end{bmatrix}:x_1^*\in\mathscr{X}_1^*(u_1^*),\ldots,x_n^*\in\mathscr{X}_n^*(u_n^*)\right\}$$

in the form

$$\bigcup_{u^*\in\mathscr{U}^*}\mathscr{X}^*(u^*).$$

THEOREM 1: *Any two-level problem derived from a solvable OCI problem is itself also solvable, and its solution is equivalent to the solution of the OCI problem:*

$$(1.30)\qquad \mathscr{U}^*\neq\emptyset\quad\text{and}\quad \mathscr{X}^*=\bigcup_{u^*\in\mathscr{U}^*}\mathscr{X}^*(u^*).$$

[13] E.g., for the case of $b^{\#}\geqslant 0$, $b_1^o>0$, $\mathscr{U}$ is a proper subset of $\dot{\mathscr{U}}$.

The maximum value of the over-all optimum is equal to the optimum of the OCI problem:

$$(1.31)\qquad \max_{u\in\mathcal{U}} \varphi(u) = \varphi(u^*) = \max_{x\in\mathcal{X}} c'x = \Phi \qquad (u^*\in\mathcal{U}^*)\,.$$

PROOF: The statements of the theorem may be read from the following:

$$\Phi = \max_{x\in\mathcal{X}} c'x = \max_{x\in \bigcup_{u\in\mathcal{U}}\mathcal{X}(u)} c'x = \max_{u\in\mathcal{U}}(\max_{x\in\mathcal{X}(u)} c'x)$$
$$= \max_{u\in\mathcal{U}}(\sum_{i=1}^{n} \max_{x_i\in\mathcal{X}_i(u_i)} c_i'x_i) = \max_{u\in\mathcal{U}} \varphi(u) = \varphi(u^*)\,.$$

1.2. *The transformation of the two-level problem into a polyhedral game, and the latter's iterative solution*

The objective function of the concave programming problem to be solved at the "central level" of the two-level problem is the over-all optimum $\varphi(u)$. This function may indeed be determined on the basis of the data of the OCI problem and its decomposition into sectors, but it is not an easy task. The two-level problem is therefore suitably transformed. Let $\mathcal{V}$ denote the set of feasible sector shadow price system teams according to (1.12):

$$(1.32)\qquad \mathcal{V} = \mathcal{Y}_1\times\ldots\times\mathcal{Y}_n = \{v = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} : y_1\in\mathcal{Y}_1, \ldots, y_n\in\mathcal{Y}_n\}\,.$$

Then the following expression is obtained for the over-all optimum:

$$(1.33)\qquad \varphi(u) = \sum_{i=1}^{n} \varphi_i(u_i) = \sum_{i=1}^{n} \min_{y_i\in\mathcal{Y}_i} y_i'u_i = \min_{\substack{y_i\in\mathcal{Y}_i \\ (i=1,\ldots,n)}} \sum_{i=1}^{n} y_i'u_i = \min_{v\in\mathcal{V}} v'u\,.$$

Hence, according to (1.31), the following may be written for the OCI optimum:

$$(1.34)\qquad \Phi = \max_{u\in\mathcal{U}} \min_{v\in\mathcal{V}} v'u\,.$$

Let us define a polyhedral game[14] in the following terms: Let $\mathcal{U}$ be the set of strategies of the maximizing, and $\mathcal{V}$ of the minimizing player, and let the homogeneous bilinear function $v'u$ ($u\in\mathcal{U}$, $v\in\mathcal{V}$) be the pay-off function of the game. The maximizing player may be identified with the "centre," the minimizing player with the "team of sectors."[15] Consequently we may speak of a central strategy in place of the feasible central program, sector strategy in place of the feasible sector shadow price system team, and in the case of both strategies we may consider the various sector components of the strategy. The game which has thus been defined

[14] See Wolfe [25]. In this case $m = n$, $X = \mathcal{U}$, $Y = \mathcal{V}$, $A = E =$ identity matrix.
[15] See the definition of a "person," e.g., in McKinsey [20, p. 4, para. 3].

is called the polyhedral game derived from the two-level problem or from the OCI problem through the given decomposition of sectors and the given choice of the set of feasible central programs, in symbols: $(\mathscr{U}, \mathscr{V})$. What the relation (1.34) expresses is that the optimum of the OCI problem is the max-min (lower) value of the polyhedral game derived from it. The connection between the OCI problem and the two-level problem and the polyhedral game derived from it is contained in the following theorem:

THEOREM 2: *A polyhedral game derived from a solvable OCI problem is itself solvable and its value equals the OCI optimum. The optimal strategies of the "central player" are the optimal central programs appearing in the corresponding two-level problem. Among the optimal strategies of the "sector-team player" there is always a strategy whose sector components are equal; to be optimal, the necessary and sufficient condition for a sector strategy whose sector components are equal is that it should be the optimal counter-strategy against some central strategy.*[16] *In an optimal sector strategy whose sector components are equal, this common sector component forms an optimal OCI shadow price system, and vice versa.*

PROOF: (1) In the case of a solvable OCI problem the OCI optimum ϕ exists and is finite, while according to (1.34) it is equal to the max-min value of the polyhedral game $(\mathscr{U}, \mathscr{V})$. From this it follows, according to the theorem of Wolfe [25], that Φ is at the same time also the min-max (upper) value of this game, so that $(\mathscr{U}, \mathscr{V})$ is solvable, and its value is Φ, and both players have optimal strategies.

(2) Since according to (1.33) $\varphi(u)$ is the minimum of the payoff function $v'u$ on the set $\mathscr{V}$, the set consisting of the optimal strategies of the central player equals the set in (1.28).

(3) Using the notation of (1.3)–(1.12) it will first be shown that

$$(1.35) \qquad \mathscr{Y}_1^*(u_1) \cap \ldots \cap \mathscr{Y}_n^*(u_n) = \begin{cases} \mathscr{Y}^*, & \text{if} \quad u \in \mathscr{U}^*, \\ \emptyset, & \text{if} \quad u \notin \mathscr{U}^*, \end{cases}$$

so that an optimal counter-strategy, the sector-components of which are equal, exists only as a counter-strategy against some optimal central strategies, but then always exists, and the common sector component is an optimal OCI shadow price system. As the first part of the proof of (1.35), it can be demonstrated that for $y^* \in \mathscr{Y}^*$, $u^* \in \mathscr{U}^*$ it is true that $y^* \in \mathscr{Y}_i^*(u_i^*)$, $i = 1, \ldots, n$. For if the opposite were the case, then we would have $\Phi = \varphi(u^*) = \sum_{i=1}^{n} \varphi_i(u_i^*) = \sum_{i=1}^{n} \min_{y_i \in \mathscr{Y}_i} y_i' u_i^* < \sum_{i=1}^{n} y^{*\prime} u_i^* =$

[16] $\hat{v} \in \mathscr{V}$ is an optimal counter-strategy against the central strategy $u \in \mathscr{U}$, *if* $\hat{v}'u = \min_{v \in \mathscr{V}} v'u = \varphi(u)$.

$y^{*\prime} \sum_{i=1}^{n} u_i^* = y^{*\prime} b = \Phi$, which is impossible. As the second part of the proof of (1.35), it can be shown that in the reverse case, for $\hat{y} \in \mathscr{Y}_i^*(\hat{u}_i)$, $i=1, \ldots, n$, it is true that $\hat{y} \in \mathscr{Y}^*$, $\hat{u} \in \mathscr{U}^*$. For let $y \in \mathscr{Y}$; then $y' b = y' \sum_{i=1}^{n} \hat{u}_i = \sum_{i=1}^{n} y' \hat{u}_i \geqslant \sum_{i=1}^{n} \min_{y_i \in \mathscr{Y}_i} y_i' \hat{u}_i = \sum_{i=1}^{n} \varphi_i(\hat{u}_i)$ $= \varphi(\hat{u}) = \sum_{i=1}^{n} \hat{y}' \hat{u}_i = \hat{y}' \sum_{i=1}^{n} \hat{u}_i = \hat{y}' b$, so that $y' b \geqslant \hat{y}' b$, i.e., $\hat{y} \in \mathscr{Y}^*$. It follows, furthermore, that $\varphi(\hat{u}) = \hat{y}' b = \Phi$, so that $\hat{u} \in \mathscr{U}^*$. (1.35) has thus been proved. To complete the proof of Theorem 2 it is now only necessary to show that for any optimal OCI shadow price system y^*, the sector strategy $\tilde{v}^* = [y^{*\prime}, \ldots, y^{*\prime}]'$ with equal sector components y^* is an optimal one. For this it is sufficient to demonstrate that $\tilde{v}^*$, together with any optimal central strategy, $u^* \in \mathscr{U}^*$, forms a saddle-point of the function $v'u$ in $(\mathscr{U}, \mathscr{V})$. For, if we let $u^* = [u_1^{*\prime}, \ldots, u_n^{*\prime}]'$ be the chosen optimal central strategy, and $u = [u_1', \ldots, u_n']' \in \mathscr{U}$ as well as $v = [y_1', \ldots, y_n']' \in \mathscr{V}$ be arbitrary, then $\tilde{v}^{*\prime} u = \sum_{i=1}^{n} y^{*\prime} u_i = y^{*\prime} \sum_{i=1}^{n} u_i = y^{*\prime} b = \Phi = y^{*\prime} \sum_{i=1}^{n} u_i^* = \sum_{i=1}^{n} y^{*\prime} u_i^* =$ $\tilde{v}^{*\prime} u^* \leqslant \sum_{i=1}^{n} y_i' u_i^* = v' u^*$, so that $\tilde{v}^{*\prime} u \leqslant \tilde{v}^{*\prime} u^* \leqslant v' u^*$ $(u \in \mathscr{U},\ v \in \mathscr{V})$. Theorem 2 is thus proved.

The OCI problem has thus been reduced to a solution of the polyhedral game $(\mathscr{U}, \mathscr{V})$ derived from it. To find a solution which would utilize the decomposition of the problem and be built up of partial calculations that can take place separately at the centre and in the sectors, we define the concept of an evaluable sector strategy (or an evaluable sector shadow price system team). This is understood to mean a sector strategy, $v \in \mathscr{V}$, against which there exists an optimal central counter-strategy; in other words, one where the linear programming problem $u \in \mathscr{U}$, $v'u \to \max!$ is solvable. Let a derived polyhedral game $(\mathscr{U}, \mathscr{V})$ be called *regular* if all its sector strategies are evaluable. *It is henceforth assumed that the polyhedral game* $(\mathscr{U}, \mathscr{V})$ *is regular.*

Consider that, according to the definition of a two-level problem, every element of $\mathscr{U}$ is an evaluable central strategy, i.e., the linear programming problem $v \in \mathscr{V}$, $v'u \to \min!$ is solvable for every $u \in \mathscr{U}$. In the case of a regular polyhedral game, therefore, there is an optimal counter-strategy against each strategy of both players. It hence follows that all regular polyhedral games are strategically reducible[17] to a matrix game.[18] Let $\mathscr{U} = U^{\triangle} + \bar{U}^{<}$ and $\mathscr{V} = V^{\triangle} + \bar{V}^{<}$ be the canonical decomposition of the strategy sets concerned. From the two kinds of solvability assumptions it immediately follows, analogously with (1.16), that $V'\bar{U} \leqslant O$, $\bar{V}'\bar{U} = O$, and $\bar{V}' U \geqslant O$, and moreover that the strategy $u = Up + \lambda \bar{U} \bar{p} \in \mathscr{U}$

[17] A game is said to be strategically reducible to another game if the latter is solvable and its every solution (optimal strategy-pair) is a solution of the original game, too.

[18] For a definition of the matrix game, see, e.g., Karlin [**10**, p. 17].

can (in the wide sense) be dominated by the strategy $u^{\triangle}=Up\in U^{\triangle}$; the strategy $v=Vq+\mu\bar{V}\bar{q}\in\mathscr{V}$, by the strategy $v^{\triangle}=Vq\in V^{\triangle}$, i.e., that for all strategies $v^0\in\mathscr{V}$, $v^{0\prime}u\leqslant v^{0\prime}u^{\triangle}$, and for all strategies $u^0\in\mathscr{U}$, $v'u^0\geqslant v^{\triangle\prime}u^0$, (where $p,\bar{p},q,\bar{q}$ are probability vectors, and λ and μ are nonnegative numbers).

The polyhedral game $(\mathscr{U},\mathscr{V})$ is therefore strategically reducible to the polyhedral game $(U^{\triangle}, V^{\triangle})$, and this in turn is isomorphic with the matrix game having a payoff matrix $V'U$. It should be pointed out that this matrix game is only stated in an implicit form since the payoff matrix $V'U$ is not directly known, the only available fact being that U and V are the matrices composed by the extreme vectors of the polyhedral sets $\mathscr{U}$ and $\mathscr{V}$, defined by the inequality systems of the centre and the sectors, respectively. If, therefore, it is intended to find a solution of the regular polyhedral game $(\mathscr{U},\mathscr{V})$ by solving the matrix game with the payoff matrix $V'U$, then for this reason itself—apart from the difficulties of a computing problem of at least the same size as the original OCI problem—it is impossible to apply direct calculation procedures.

It is possible, on the other hand, to use the fictitious play method of Brown and Robinson.[19] According to this method, against each central strategy $u\in\mathscr{U}$ of the regular polyhedral game $(\mathscr{U},\mathscr{V})$ it is possible to state an optimal counter-strategy $v^*(u)\in V^{\triangle}$, while against each sector strategy $v\in\mathscr{V}$ it is possible to state an optimal counter-strategy $u^*(v)\in\mathscr{U}^{\triangle}$, for which

$$(1.36)\qquad v^*(u)'u=\min_{v\in\mathscr{V}} v'u\,,\qquad v'u^*(v)=\max_{u\in\mathscr{U}} v'u$$

holds. The term $v^*(u)$ is called the *regular evaluation* of u, $u^*(v)$ of v. The *regular fictitious play* of the regular polyhedral game $(\mathscr{U},\mathscr{V})$ is understood to mean the construction according to the rule stated below of the strategy series $u^*\langle 1\rangle$, $u^*\langle 2\rangle,\ldots,u^*\langle N\rangle,\ldots$ within $U^{\triangle}$, and $v^*\langle 1\rangle$, $v^*\langle 2\rangle,\ldots,v^*\langle N\rangle,\ldots$ within $V^{\triangle}$.

Phase 1. Step I. Select any central strategy $u^{(1)}\in U^{\triangle}$.

Step II. By definition $u^*\langle 1\rangle=u^{(1)}$.

Step III (The regular evaluation of $u^*\langle 1\rangle$). The determination of $v^{(1)}=v^*(u^*\langle 1\rangle)$.

Step IV. By definition, $v^*\langle 1\rangle=v^{(1)}$.

The process goes on for phases 2, 3, ...

Phase N ($N=2,3,\ldots$). Step I (The regular evaluation of $v^*\langle N-1\rangle$). The determination of $u^{(N)}=u^*(v^*\langle N-1\rangle)$.

Step II ("Mixing" with the term of the previous phase). The calculation of $u^*\langle N\rangle=[(N-1)/N]\; u^*\langle N-1\rangle+(1/N)\; u^{(N)}$.

Step III (The regular evaluation of $u^*\langle N\rangle$): The determination of $v^{(N)}=v^*(u^*\langle N\rangle)$.

Step IV ("Mixing" with the term of the previous phase): The calculation of $v^*\langle N\rangle=[(N-1)/N]\; v^*\langle N-1\rangle+(1/N)\; v^{(N)}$.

As $\Phi=\max_{u\in\mathscr{U}}\min_{v\in\mathscr{V}} v'u=\min_{v\in\mathscr{V}}\max_{u\in\mathscr{U}} v'u$, it can easily be deduced from the

[19] Brown [**1, 2**]; Robinson [**21**]. For a detailed discussion: Karlin [**10**, pp. 179–189].

definitions of the evaluations that the upper optimum $\Phi^*\langle N\rangle = \max_{u\in\mathcal{U}} v^*\langle N-1\rangle' u = \max_{u\in\mathcal{U}\triangle} v^*\langle N-1\rangle' u = v^*\langle N-1\rangle' u^{(N)}$ in the Nth phase ($N=2, 3, \ldots$) supplies an upper estimate for the OCI optimum Φ, the lower optimum $\varphi^*\langle N\rangle = \min_{v\in\mathcal{V}} v' u^*\langle N\rangle = \min_{v\in\mathcal{V}\triangle} v' u^*\langle N\rangle = v^{(N)\prime} u^*\langle N\rangle$ in the Nth phase ($N=1, 2, 3, \ldots$), a lower estimate for it, i.e.,

(1.37) $\quad \Phi^*\langle N\rangle \geqslant \Phi \quad (N=2, 3, \ldots); \quad \varphi^*\langle N\rangle \leqslant \Phi \quad (N=1, 2, 3, \ldots).$

Since, moreover, the series also imply, on account of their construction, the fictitious play of the matrix game with the payoff matrix $V'U$, the Brown-Robinson theorem is valid, so that

(1.38) $\quad \lim_{N\to\infty} \Phi^*\langle N\rangle = \lim_{N\to\infty} \Phi^*\langle N\rangle = \Phi,$

and the limit points of the series $\{u^*\langle N\rangle\}$ and $\{v^*\langle N\rangle\}$ are optimal central and sector strategies.

The δ-termination of the iteration of regular fictitious play (where δ is an arbitrary small positive number) is understood to mean the following termination of the above construction: Let N_δ be the least positive integer for which

(1.39) $\quad \Phi^*\langle N_\delta\rangle - \varphi^*\langle N_\delta\rangle \leqslant \delta \quad$ or $\quad \Phi^*\langle N_\delta+1\rangle - \varphi^*\langle N_\delta\rangle \leqslant \delta$

holds. According to (1.37) and (1.38), N_δ may be defined for an arbitrarily small positive number δ. Then: (1) The iteration is terminated at Step II of phase N_δ, or at Step I of phase $(N_\delta+1)$ (according to whether the first or the second inequality in (1.39) has been satisfied); (2) The linear programming problems

(1.40) $\quad A_i x_i \leqslant u_i^*\langle N_\delta\rangle, \qquad x_i \geqslant 0, \qquad c_i' x_i \to \max! \qquad (i=1, \ldots, n)$

in the sectors are solved; (3) From the sector programs $x_i^*(u_i^*\langle N_\delta\rangle) = x_i^{\delta*}$ thus obtained, the feasible OCI program $x^{\delta*} = [x_1^{\delta*\prime}, \ldots, x_n^{\delta*\prime}]'$ is composed. Since $c'x^{\delta*} = \sum_{i=1}^{n} c_i' x_i^{\delta*} = \sum_{i=1}^{n} y_i^*(u_i^*\langle N_\delta\rangle)' u_i^*\langle N_\delta\rangle = v^*(u^*\langle N_\delta\rangle)' u^*\langle N_\delta\rangle = \varphi^*\langle N_\delta\rangle$, then, acording to (1.39),

(1.41) $\quad \Phi - \delta \leqslant c'x^{\delta*} \leqslant \Phi.$

(1.41) is briefly referred to as the fact that $x^{\delta*}$ is a δ-optimal OCI program.

As a supplement, take the case where the polyhedral game $(\mathcal{U}, \mathcal{V})$ can be solved uniquely for the sector-team player. It follows that its reduced version, the matrix game with the payoff matrix $V'U$, also possesses this property, so that according to the previously quoted Brown-Robinson theorem, the series $\{v^*\langle N\rangle\}$ is convergent, and its limit is the unique optimal sector strategy. On the basis of part 3 of Theorem 2, this is none other than that sector shadow price system team whose every sector component is the (in consequence of the above conditions) unique, optimal OCI shadow price system. Symbolically: $\lim_{N\to\infty} v^*\langle N\rangle = [y^{*\prime}, \ldots, y^{*\prime}]$, where $y^{*\prime}b = \min_{y\in\mathcal{Y}} y'b$,

The results pertaining to the fictitious play of the polyhedral game derived from the OCI problem, whose proofs have been furnished above, may be summarized as follows:

THEOREM 3: *In the case of a regular polyhedral game derived from a solvable OCI problem, the latter may be solved through regular fictitious play to any required degree of accuracy, in the sense that for an arbitrarily small positive δ the δ-termination of regular fictitious play leads to a δ-optimal OCI program. If, at the same time, the derived regular polyhedral game can be uniquely solved for the sector-team player, the sector components of the mixed sector strategy series obtained in the course of regular fictitious play are equalized, i.e., they converge towards a common limit which is the optimal OCI shadow price system.*

1.3. *Supplementary comments*

The OCI problem must be transformed into a regular polyhedral game. This is done by decomposition into sectors and the choice of a suitable generating set consisting of evaluable central programs. It has not yet been discussed whether this transformation can be carried out for all linear programming problems —regarded as OCI problems—and, if a problem can be transformed, which of the several kinds of transformation it is best to choose. Nor has the problem been examined of whether computing technique can deal with the central programming problem in Step I of the iterative phases, for in the general case the number of variables in the central programming problem, i.e., the number of components in the central program, is the product of the number of OCI conditions and the number of sectors.

It should be pointed out that in the special case where the elements of the matrix A and the bounding vector b are nonnegative numbers, under any decomposition the evaluable central programs are the vectors whose components form a non-negative partition of b, so that the set of evaluable central programs is a non-void, bounded convex polyhedral set. Since it follows that all feasible sector shadow price system teams are evaluable, any decomposition and the choice of a generating set consisting of all the evaluable central programs leads to a regular polyhedral game. Moreover, the central programming in Step I of the iteration phases of the fictitious play decomposes into "microprogramming," in the course of which the full bounds of each OCI condition are partitioned to the sector which possessed the largest shadow price referring to this condition.

Similar results are also derived in the course of the transformation of the concrete model in hand, that of a long-term macroeconomic planning problem. For this reason, the problem raised in the case of the general model is in this paper left open, and the necessary change will only be made on the concrete model.[20]

[20] Further problems relating to the general model are treated in a paper by Lipták **[18]**.
During the editorial preparation of this paper, Professor J. F. Benders was kind enough to call our attention to his paper "Partitioning of Mixed Variables Programming Problems" (in *Numerische Mathematik*, 4, 1962, pp. 238–252.) Theorem 1 (dealing with the connection between the OCI problem and the two-level problem) and Theorem 2 (dealing with the connection between the OCI problem, the two-level problem, and the derived polyhedral game) of this paper are related to the "partitioning theorem" in his paper, though they are not equivalent to it.

2. THE CONCRETE MODEL: A LONG-TERM MACROECONOMIC PLANNING PROBLEM

2.1. *Description of the model*

While in the general model the point of departure was the OCI problem, in the case of the concrete model it is more convenient to begin the discussion with the form subsequent to the decomposition into sectors. We shall therefore proceed straightaway to write down the two-level problem derived from the OCI problem. Both the criterion (1.14) and the simplifying remarks (1.20)–(1.27) were applied in formulating the set of central programs. The various components of the central program and the sector programs are, in agreement with their economic nature, denoted by different letters, while their sign has been chosen for ease of notation.

Planning is directed by the centre—in actual practice by the National Planning Bureau. There are altogether n sectors. Each sector is responsible for a particular group of products; in the subsequent discussion the term products will be used for *product-groups*, for the sake of brevity. The activities of the sector comprise not only the domestic production of the product concerned and the investments necessary for production, but also the export and import of the product. A long-term plan is to be worked out for a plan-term, consisting of altogether T periods.

This model is not meant to determine all the targets of the national plan. The point of departure is a national economic plan that has already been elaborated (by "traditional," non-mathematical means, checked with an input-output table). Certain targets of this plan are adopted as constants in the programming table. In this paper they are called *economic policy figures.*

The centre issues three kinds of directives to the sectors:

(1) The centre tells the ith sector to provide a certain quantity of product to meet domestic requirements in the tth period. This quantity, whose symbol is r_{it}, is called the supply assignment ($i=1, \ldots, n$; $t=1, \ldots, T$). The centre does not prescribe whether the required quantity should be met from domestic production or imports—this will be determined by the sector program. Furthermore, it is the sector program that must determine whether the sector, beyond satisfying domestic requirements, also wishes to export.

(2) The centre assigns to the ith sector a certain quantity of the jth product for the tth period. This is symbolized by z_{ijt} and is called the materials quota ($i=1, \ldots, n$; $j=1, \ldots, n$; $j \neq i$; $t=1, \ldots, T$). The materials quota comprises the jth material derived both from home production and imports.

(3) The centre assigns a certain complement of manpower to the ith sector for the tth period. This is symbolized by w_{it} and is called the manpower quota.

The directives are the variables of the central program. The constants in the constraint system of this central program are economic policy figures. These are:

(1) Q_{it}, which stands for the external consumption of the ith product necessary in the tth period. This comprises consumption by individuals and public bodies, including nonproductive investments. On the other hand, it does not include either

exports or—apart from certain exceptions—productive investments. (The exceptions will be treated later.)

(2) R_{it}, which is the bound of the ith supply assignment in the tth period. (This bound has no real economic meaning; its introduction is only necessary for the mathematical algorithm, but there is no practical difficulty about determining the quantity which the supply assignment is sure not to exceed.)

(3) W_t, which is the manpower quota available for productive work in the economy in the tth year.

Those central programs will be considered feasible for which

$$\sum_{\substack{j=1\\ j\neq i}}^{n} z_{jit}+Q_{it}=r_{it}\leqslant R_{it} \qquad (i=1,\ldots,n;\ t=1,\ldots,T), \tag{2.1}$$

$$\sum_{i=1}^{n} w_{it}=W_t \qquad (t=1,\ldots,T) \tag{2.2}$$

$$r_{it}\geqslant 0,\quad z_{ijt}\geqslant 0,\quad w_{it}\geqslant 0 \quad (i=1,\ldots,n;\ j=1,\ldots,n;\ j\neq i;\ t=1,\ldots,T). \tag{2.3}$$

The variables in the programming model of the ith sector may be classified into several groups according to their economic nature:

(1) *Reproductive activities.* These consist of the unchanged, continued operation of the output capacities for the ith product which already existed at the beginning of the plan-term. Several kinds of these activities may be incorporated in the model according to their technical features (e.g., backward or advanced factories). Let x_{ikt} denote the level of the kth reproductive activity planned for the ith sector in the tth period:[21] ($x_{ikt}\geqslant 0$, $k=$repr[22], $t=1,\ldots,T$).

(2) *Investment activities.* This concept includes both the establishment of new capacities and the production in these new facilities. Several types of investment activity may be incorporated in the model, on the basis of technical or economic features (e.g., the technology used, whether the machinery is imported or domestic-made, etc.). Moreover, within a particular type of investment activity (e.g., the establishment and operation of a particular plant in a certain way), several kinds of investment activity may be distinguished according to the period in which the investment is begun. A separate investment variable will correspond to each of

[21] Here, and also in the case of the other variables (except for investment activities), the unit of measurement for the level of activity is the quantity of the product stated in the natural units best suited for its measurement per unit period of time, or else in forint per unit period. It must be identical with the unit of measurement used for the ith product in the corresponding central product balance according to (2.1).

[22] Neither here nor in the other groups of sector activities will the numbers of the activities be stated. Instead, a suitable abbreviation after the suffix k will indicate the character of the activity concerned, e.g., k = repr, k = inv, etc.

these alternatives. Let x_{ik} denote the level of the kth investment activity of the ith sector[23] ($x_{ik} \geqslant 0$, $k = \text{inv}$).

(3) *Export activities.* Several kinds of export activities can appear in the model, according to their economic features (e.g., by markets, countries, etc.). Let x_{ikt} denote the level of the kth export activity in the case of the ith product in the tth period ($x_{ikt} \geqslant 0$, $k = \text{exp}$, $t = 1, \ldots, T$).

(4) *Bounded import activities.* This group comprises only import activities which compete with the domestic output activities belonging to Groups 1 and 2 and are able to replace the latter and whose level is bounded by some external market factor. Several kinds of activities can figure in the model, according to their economic features (e.g., markets, etc.). Let x_{ikt} denote the level of the kth bounded import activity in the case of the importation of the ith product in the tth period ($x_{ikt} \geqslant 0$, $k = \text{imp}$, $t = 1, \ldots, T$).

(5) *Unbounded import activity.* This is an import activity that, like the import activities of Group 4, competes with domestic production, but its level is not bounded either by extraneous market factors or by other influences. In some sectors there is justification for presuming that an unbounded import activity is a realistic proposition. In other sectors no such free, unbounded import activity actually exists. The method requires that this type of variable be used even in these latter sectors as an auxiliary variable, but the programming procedure used will automatically eliminate them from the program. Let the level of unbounded import in the ith sector be denoted by x_{i0} ($x_{i0} \geqslant 0$).

The conditions prescribed for the ith sector program may be classified into one of two main groups. One group of conditions ensures that the sector should obey the directives received from the centre. The first condition is that

$$\text{(2.4)} \qquad r_{it} \leqslant \sum_{\substack{k=\text{repr, exp,}\\ \text{imp}}} f_{ikt} x_{ikt} + f_{i0t} x_{i0} + \sum_{k=\text{inv}} f_{ikt} x_{ik} \leqslant R_{it} \qquad (t = 1, \ldots, T).$$

The output coefficient f_{ikt} in this condition is as follows for the various sector activities:

(1) For reproductive activities, $f_{ikt} = 1$. (2) For investment activities, $f_{ikt} \geqslant 0$, but for at least one t, $f_{ikt} = 1$. As a result of unit investment activity there will sometime, but at the latest during the last period, come to be established a capacity unit which will permit production of a unit quantity of the kth product during one period. The preceding output on the other hand will depend on when the investment is begun, and on the amount of "turning up" required before it achieves normal operation. It is assumed that a specific time-distribution of output (and as

[23] Since, according to the above, an investment activity refers not to particular periods but is a series of investment activities over the full plan-term, the level x_{ik} is distinguished from the other variables in that it does not include the subscript t. The level of an investment activity is accordingly measured by the quantity of product which the facility that has been established produces when it is operated at full capacity, in terms of natural units or forint per unit plan period.

we shall see, of expenditures) is characteristic of the kth investment activity. It is also assumed that the capacities created by means of the investment will, after "turning up," always be utilized to the normal extent. If, therefore f_{ikt} is for some value of t equal to 1, then it is also 1 for the periods $(t+1)$, $(t+2)$, etc. In this sense then, this group of activities differs from the reproductive ones in that there is no assumption that the latter's existing old capacities must necessarily be utilized fully. (3) In the case of export activities, $f_{ikt}=-1$. (4 and 5) For bounded and unbounded import activities, $f_{ikt}=1$.

The next series of conditions linked to the central directives is that

$$(2.5)\qquad \sum_{\substack{k=\text{repr, exp,}\\ \text{imp}}} g_{ijkt}x_{ikt}+g_{ij0t}x_{i0}+\sum_{k=\text{inv}} g_{ijkt}x_{ik}\leqslant z_{ijt}\quad \begin{array}{l}(j=1,\ldots,n,\ j\neq i;\\ t=1,\ldots,T)\,.\end{array}$$

The input coefficient g_{ijkt} is the following for the various sector activities:

(1) For reproductive activities, $g_{ijkt}\geqslant 0$. Production, through its technological character, either requires or does not require the jth material. These materials requirements comprise both that of current operation and also that of upkeep of the old facilities: for general repair, replacement, and renovation necessary to ensure simple reproduction. (2) In the case of investment activities, $g_{ijkt}\geqslant 0$. This comprises the products (e.g., machines) required by investment during the years when the new facility is established, and the products necessary during the years of operation both for current production and for the maintenance and replacement of the facilities. As in the case of output, it is here again assumed that the kth investment activity is characterized by a certain distribution of material requirements in time. (3, 4, and 5) For all foreign trade activity, $g_{ijkt}=0$.

Finally, the last condition linked to the central directive is that

$$(2.6)\qquad \sum_{\substack{k=\text{repr, exp,}\\ \text{imp}}} h_{ikt}x_{ikt}+h_{i0t}x_{i0}+\sum_{k=\text{inv}} h_{ikt}x_{ik}\leqslant w_{it}\qquad (t=1,\ldots,T)\,.$$

The manpower-coefficient h_{ikt} is as follows for the various groups of activity:

(1) In the case of reproductive activity, $h_{ikt}>0$, for without manpower there can be no production. (2) For investment activities, $h_{ikt}\geqslant 0$. Before operation is begun it is 0; later it is positive—the development of its numerical value is a characteristic function of time. (3, 4, and 5) In foreign trade activities, $h_{ikt}=0$.

Beyond the conditions to secure observance of the central directives, special condition characteristics of the particular circumstances of the sector may also be stipulated. By way of example, reproductive activities might be bounded by the upper limit of existing facilities. Certain investment activities, e.g., the modernization of existing plants, might be limited. Output in some sectors might be bounded by the country's natural resources. Certain export and import activities might be limited by market factors, etc. The special conditions may be written in the following general form:[24]

$$\text{(2.7)} \qquad \sum_{t=1}^{T} \sum_{\substack{k=\text{repr, exp}\\ \text{imp}}} a^0_{iskt} x_{ikt} + \sum_{k=\text{inv}} a^0_{isk} x_{ik} \leqslant b^0_{is}, \qquad s=\text{spec.}$$

Because of the artificial character of unbounded import activities, the coefficients of these variables are 0 in all the above special conditions. We assume that the constraints (2.7) can be satisfied, so that we write 0 in place of the variables figuring in them; that is, $b^0_{is} \geqslant 0$ for every s.

The aim of the ith sector's programming is that

$$\text{(2.8)} \qquad \sum_{t=1}^{T} \sum_{\substack{k=\text{repr, exp}\\ \text{imp}}} c_{ikt} x_{ikt} + c_{i0} x_{i0} + \sum_{k=\text{inv}} c_{ik} x_{ik} \to \max!\,,$$

i.e., the maximization of the sectoral objective function on the left hand side of (2.8). Here c_{ikt} and c_{ik} are the foreign exchange returns of the corresponding activity. These are the following for the various groups of activities:

(1 and 2) For reproductive and investment activities the foreign currency returns are generally zero. An exception is formed by the production and investment activities, which require noncompetitive imports that cannot be satisfied by home production. Noncompetitive import costs are regarded as negative foreign currency returns. (In the case of investment activities, the foreign currency returns naturally comprise all the noncompetitive import costs incurred throughout the plan-term.) (3) The foreign currency returns of export activities are positive. (4 and 5) The foreign currency returns of import activities are negative. If unbounded imports are only a fictitious variable, they are weighted with very heavy negative foreign currency returns. It is also assumed with respect to the yield of foreign trade activities that

$$\text{(2.9)} \qquad \max_{k=\text{exp}} c_{ikt} \leqslant \min\left\{\min_{k=\text{imp}} (-c_{ikt}), -c_{i0}\right\} \qquad (t=1, \ldots, T)\,.$$

On the macroeconomic scale, that central program is regarded as optimal under which the sum of the maximal values of the sectoral objective functions is maximal.

In the dual of the ith primal sector problem according to (2.4)–(2.8) and under the central program (directives) denoted by $(r_{it}, z_{ijt}, w_{it})$, let ρ_{it} be the shadow price of the central directive r_{it}; ζ_{ijt} of z_{ijt}; ω_{it} of w_{it}; while π_{it} and σ_{is} will be the shadow prices of the special conditions bounded by R_{it} and b^0_{is}. Then with the sector shadow price system $(\rho_{it}, \zeta_{ijt}, \omega_{it}, \pi_{it}, \sigma_{is})$ as the dual program, the ith dual sector problem has the following form:

[24] The abbreviation s = spec. expresses the fact that all the special conditions are enumerated in (2.7). See footnote 22 on p. 157.

$$(2.10)\quad f_{ikt}(\pi_{it}-\rho_{it}) + \sum_{\substack{j=1\\ j\neq i}}^{n} g_{ijkt}\zeta_{ijt}+h_{ikt}\omega_{it}+\sum_{s=\text{spec}} a^0_{iskt}\sigma_{is}\geqslant c_{ikt}$$
$$k=\text{repr, exp, imp}\qquad (t=1,\ldots,T),$$

$$(2.11)\quad f_{ikt}(\pi_{it}-\rho_{it}) + \sum_{j=1}^{n} g_{ijkt}\zeta_{ijt}+h_{ikt}\omega_{it}+\sum_{s=\text{spec}} a^0_{isk}\sigma_{is}\geqslant c_{ik}$$
$$k=0,\text{inv}\qquad (t=1,\ldots,T),$$

$$(2.12)\quad \rho_{it}\geqslant 0,\ \zeta_{ijt}\geqslant 0,\ \omega_{it}\geqslant 0,\ \pi_{it}\geqslant 0,\ \sigma_{is}\geqslant 0,\quad (j=1,\ldots,n,\ j\neq i;\ t=1,\ldots,T,$$

$$(2.13)\quad \sum_{t=1}^{T}(R_{it}\pi_{it}-r_{it}\rho_{it}+\sum_{\substack{j=1\\ j\neq i}}^{n} z_{ijt}\zeta_{ijt}+w_{it}\omega_{it})+\sum_{s=\text{spec}} b^0_{is}\sigma_{is}\to\min!$$

It will be left to the reader to check that the OCI problem corresponding to this two-level problem is solvable, and that the polyhedral game derived from it is a regular one. (In the latter case the criterion (1.14) and the boundedness of the set of feasible central programs may be used.) In the following, the iterative procedure of the δ-termination of the regular fictitious play, i.e., of the construction of a δ-optimal macroeconomic program will be outlined (where δ is an arbitrary small positive number).

2.2. *The process of iteration*

Since the set of feasible central programs is here bounded, any central program $(r_{it}^{(1)}, z_{ijt}^{(1)}, w_{it}^{(1)})$ may be made the starting point.[25] Passing over the first phases, we come to the Nth phase. From the previous $(N-1)$st phase: (1) the "central memory" stores the upper optimum $\Phi^*\langle N-1\rangle$, the central program $(r^*_{it}\langle N-1\rangle$, $z^*_{ijt}\langle N-1\rangle, w^*_{it}\langle N-1\rangle)$ sent down to the sectors, the sector shadow prices $\rho^*_{it}\langle N-1\rangle, \zeta^*_{ijt}\langle N-1\rangle, \omega^*_{it}\langle N-1\rangle$ sent up from the sectors, the sector optima $\varphi_i^{(n-1)}$, and the mixed special sector optimum components $\phi_i^0\langle N-1\rangle$; (2) all the "sector memories" store the shadow prices $\rho^*_{it}\langle N-1\rangle, \zeta^*_{ijt}\langle N-1\rangle, \omega^*_{it}\langle N-1\rangle$ sent up to the centre, and the simplex tableau and optimal basis used in computing the provisional sector shadow prices $\varrho_{it}^{(N-1)}, \zeta_{ijt}^{(N-1)}, \omega_{it}^{(N-1)}$. (The terms and symbols enumerated will be defined below in the course of the analogous terms and symbols of the Nth phase.)

Step I. Examination of whether the iteration can be terminated and the evaluation of the sector shadow prices that are sent up, *at the centre*: The formula $\varphi^*\langle N-1\rangle=\sum_{i=1}^{n}\varphi_i^{(N-1)}$ is used to calculate the lower optimum of the previous phase, and this is compared with the upper optimum, $\Phi^*\langle N-1\rangle$, stored in the memory.

[25] For in the case of a bounded $\mathscr{U}$, then $U^{\triangle}=\mathscr{U}$. See Goldman [7, p. 49, Corollary 1B].

Case 1. $\Phi^*\langle N-1\rangle - \varphi^*\langle N-1\rangle \leqslant \delta$: $N_\delta = N-1$. The iteration is terminated. The sectors are instructed to compute the primal optimal sector programs corresponding to the provisional sector shadow prices $\rho_{it}^{(N-1)}$, $\zeta_{ijt}^{(N-1)}$, $\omega_{it}^{(N-1)}$, using the simplex tableau and optimal basis stored in their memories.[26] These sector programs constitute a δ-optimal program of the long-term macroeconomic planning problem.

Case 2. $\Phi^*\langle N-1\rangle - \varphi^*\langle N-1\rangle > \delta$: $N_\delta > N-1$. The central evaluation of the sector shadow prices that are sent up is performed, i.e., the central objective function

$$\sum_{i=1}^{n}\sum_{t=1}^{T}\Big(\rho_{it}^*\langle N-1\rangle r_{it} + \sum_{\substack{j=1\\ j\neq i}}^{n} \zeta_{ijt}^*\langle N-1\rangle z_{ijt} + \omega_{it}^*\langle N-1\rangle w_{it}\Big)$$

is maximized under the conditions (2.1)–(2.3). This linear programming problem decomposes into two types of simply solvable "micro-programming" problems: (i) the programming problem

$$\text{(2.14)}\quad \begin{cases} \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n} z_{jit} + Q_{it} = r_{it} \leqslant R_{it}\,,\ z_{jit} \geqslant 0\ \ (j=1,\dots,n,\ j\neq i;\ r_{it}\geqslant 0)\,, \\ \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n} \zeta_{jit}^*\langle N-1\rangle z_{jit} - \rho_{it}^*\langle N-1\rangle \delta_{it} \to \max!\,, \end{cases}$$

which yields the provisional production and distribution of the ith product in the tth period ($i=1,\dots,n;\ t=1,\dots,T$); (ii) the programming problem

$$\text{(2.15)}\quad \begin{cases} \displaystyle\sum_{i=1}^{n} w_{it} = W_t\,, \\ \quad w_{it} \geqslant 0 \qquad (i=1,\dots,n)\,, \\ \displaystyle\sum_{i=1}^{n} \omega_{it}^*\langle N-1\rangle w_{it} \to \max!\,, \end{cases}$$

which formulates the provisional distribution of the economy's manpower quota in the tth period ($t=1,\dots,T$).

A solution of (2.14) is as follows:

(a) if $\max_{j\neq i} \zeta_{jit}^*\langle N-1\rangle < \rho_{it}^*\langle N-1\rangle$, then

$$\text{(2.16)}\qquad r_{it}^{(N)} = Q_{it},\quad z_{jit}^{(N)} = 0 \qquad (j=1,\dots,n, j\neq i);$$

[26] The components of the optimal primal program may be directly read from the values of the "z-row", corresponding to the slack variables of the simplex tableau in the dual problem. See, for instance, Karlin [**10**, pp. 169–170].

(b) if $\max_{j \neq i} \zeta^*_{jit}\langle N-1\rangle = \zeta^*_{j_0 it}\langle N-1\rangle \geqslant \rho^*_{it}\langle N-1\rangle$,[27] then

$$(2.17) \qquad r_{it}^{(N)} = R_{it}, \quad z_{jit}^{(N)} = \begin{cases} R_{it} - Q_{it}, & \text{if } j = j_0\,, \\ 0 \quad , & \text{if } j \neq j_0\,. \end{cases}$$

The solution of (2.15) is still simpler: if $\max_i \omega^*_{it}\langle N-1\rangle = \omega^*_{i_0 t}\langle N-1\rangle$[28], then

$$(2.18) \qquad w_{it}^{(N)} = \begin{cases} W_t, & \text{if } i = i_0\,, \\ 0\,, & \text{if } i \neq i_0\,. \end{cases}$$

It is not difficult to see that the sum of the maxima derived in the course of the micro-programs (2.14)–(2.15) is

$$(2.19) \qquad \begin{aligned} \sum_{t=1}^{T} \Bigg\{ \sum_{i=1}^{n} \Big[&(\max_{j \neq i} \zeta^*_{jit}\langle N-1\rangle - \rho^*_{it}\langle N-1\rangle)^+ \cdot (R_{it} - Q_{it}) \\ &- \rho^*_{it}\langle N-1\rangle \cdot Q_{it} \Big] + \max_i \omega^*_{it}\langle N-1\rangle \cdot W_t \Bigg\} \\ &= \Phi^{\#}\langle N\rangle, \text{ say.}^{29} \end{aligned}$$

Hence the upper optimum in the Nth phase may be computed from the formula

$$(2.20) \qquad \phi^*\langle N\rangle = \phi^{\#}\langle N\rangle + \sum_{i=1}^{n} \Phi_i^0\langle N-1\rangle\,.$$

Next a comparison is made of the values $\Phi^*\langle N\rangle$ and $\varphi^*\langle N-1\rangle$.

Case 2/1. $\Phi^*\langle N\rangle - \varphi^*\langle N-1\rangle \leqslant \delta$: $N_\delta = N-1$. The iteration is terminated, and the δ-optimal macroeconomic program is computed in a fashion identical with Case 1.

Case 2/2. $\Phi^*\langle N\rangle - \varphi^*\langle N-1\rangle > \delta : N_\delta > N-1$. There then follows:

Step II. Making a new *central* program to be sent down to the sectors:

$$(2.21) \qquad \begin{cases} r^*_{it}\langle N\rangle = \dfrac{N-1}{N}\, r^*_{it}\langle N-1\rangle + \dfrac{1}{N}\, r_{it}^{(N)}\,, \\ z^*_{ijt}\langle N\rangle = \dfrac{N-1}{N}\, z^*_{ijt}\langle N-1\rangle + \dfrac{1}{N}\, z_{ijt}^{(N)}\,, \\ w^*_{it}\langle N\rangle = \dfrac{N-1}{N}\, w^*_{it}\langle N-1\rangle + \dfrac{1}{N}\, w_{it}^{(N)} \quad (j=1, \ldots, n;\ j \neq i; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t = 1, \ldots, T). \end{cases}$$

[27] If the shadow price $\zeta^*_{jit}\langle N-1\rangle$ is equally maximal in several sectors j (with i and t fixed), and if it is not less than $\varrho^*_{jt}\langle N-1\rangle$, then $R_{it} - Q_{it}$ may be partitioned between these sectors in any proportion.

[28] If the shadow price $\omega^*_{it}\langle N-1\rangle$ is equally maximal in several sectors i (with t fixed), then W_t may be partitioned between them in any proportion.

[29] α^+ is used to denote the positive part of α: $\alpha^+ = \max(\alpha, 0)$.

Step III. The evaluation *in the sectors* of the central program component that has newly been sent down. The linear programming problems (2.10)–(2.13) are solved in the ith sector, with the new objective function coefficients $r^*_{it}\langle N\rangle$, $z^*_{ijt}\langle N\rangle$, $w^*_{it}\langle N\rangle$. The simplex tableau and optimal base[30] stored in the memory may be used for the calculation. Let the solutions $\rho^{(N)}_{it}$, $\zeta^{(N)}_{ijt}$, $\omega^{(N)}_{it}$, $\pi^{(N)}_{it}$, $\sigma^{(N)}_{is}$ thus obtained be called the *provisional sector shadow prices*; let the minimal value $\varphi^{(N)}_i$ of the corresponding objective function according to (2.13) be called the *sector optimum*; and let the portion

$$\Phi^{0(N)}_i = \sum_{t=1}^{T} R_{it}\pi^{(N)}_{it} + \sum_{s=\text{spec}} b^0_{is}\sigma^{(N)}_{is} \tag{2.22}$$

of the optimum concerned with the special shadow prices be called the *special sector optimum component.*

Step IV. Preparing *in the sectors* the new sector shadow prices and mixed special sector optimum components to be sent up to the centre. In the ith sector it is necessary to compute

$$\left\{\begin{aligned} \rho^*_{it}\langle N\rangle &= \frac{N-1}{N}\rho^*_{it}\langle N-1\rangle + \frac{1}{N}\rho^{(N)}_{it}, \\ \zeta^*_{ijt}\langle N\rangle &= \frac{N-1}{N}\zeta^*_{ijt}\langle N-1\rangle + \frac{1}{N}\zeta^{(N)}_{ijt}, \quad \begin{array}{l}(j=1,\ldots,n,\ j\neq i;\\ t=1,\ldots,T),\end{array} \\ \omega^*_{it}\langle N\rangle &= \frac{N-1}{N}\omega^*_{it}\langle N-1\rangle + \frac{1}{N}\omega^{(N)}_{it} \end{aligned}\right. \tag{2.23}$$

and the mixed special sector optimum component

$$\Phi^0_i\langle N\rangle = \frac{N-1}{N}\Phi^0_i\langle N-1\rangle + \frac{1}{N}\Phi^{0(N)}_i, \qquad \Phi^0_i\langle 1\rangle = \Phi^{0(1)}_i. \tag{2.24}$$

When these—together with the sector optimum $\varphi^{(N)}_i$—are sent up to the centre, the Nth iterative phase is completed.

2.3. *On the economic interpretation of the model*

The concrete model will now be discussed from the economic point of view. An attempt will be made to interpret some features and properties of the model in terms of economics, and some problems in ascertaining the parameters figuring in the model will be raised.

(1) What does the objective function of the dual problem of the sector models express in terms of economics? Let us presume for a moment that the centre really lets the sector have its resources at a "price" corresponding to the shadow price which the sector reports back, and that at the same time it demands of the sector that it should not operate at a loss. If the sector reported back too high, "rosy"

[30] See, for instance, Suzuki [23, pp. 95–96].

shadow prices (e.g., if it stated that a rise in the manpower quota would secure greater surplus returns than it is actually capable of, according to the optimal program), then the sector would operate at a loss. The minimization of the evaluation of the boundary conditions in terms of shadow prices, as an optimization requirement of the model and the nonprofit conditions in the constraints of the dual model express the fact that care must be taken to avoid over-estimating the modifications in the central directives which appear as limitations in the sector conditions, and to avoid over-estimating the effect of these modifications on the objective function. The minimization of the dual objective function expresses an approach of careful, responsible moderation in determining the indices of economic efficiency presented as part of the report back.

(2) It is a noteworthy fact that in the case of a macroeconomic planning model the game-theoretical model may be realistically interpreted and invested with economic significance. The situation shows analogy with strategic games in that each player is in possession of certain information, but neither can make fully satisfactory decisions without obtaining some information from the other player. The centre has a broad purview, but it has no detailed knowledge of the special problems that are known to the sectors (e.g., the technical and cost figures for the various sectors, the special conditions limiting choice within the sector, etc.). The sectors see many details, but they have no ability to survey the great interrelations that can only be clear to the centre. Just as in strategic games, the situation which evolves depends on both players. Both the centre and the sectors clearly know that the measures employed by the other player exercise a great influence on the situation. Under such circumstances both players seek the relatively most reassuring strategy for themselves. This strategy is the "minimax" solution of the game.

In the present model the acceptance of the minimax strategy means the following: Let us presume that the centre is "omniscient," is in possession of even those special detailed items of information that are usually only known accurately to the sectors. In this case (if ideal computing facilities were available), it would be able to elaborate the optimal program for the national economy (the optimal OCI program). The program thus determined would have a certain objective function value and result in optimal economic returns (the OCI optimum).

If the centre (both in this model and in real life) lacks information, it will be unable to determine, without the collaboration of the sectors, the optimal program for the economy and the optimal value will not be achieved. The consequence of decisions taken independently of the sectors would be relative losses, which the centre must strive to cut.

On the other hand, the sectors, without the directing and coordinating activity of the centre, will necessarily furnish a faulty evaluation of the resources and quotas allocated to them and cannot achieve the optimal program for the economy. Let us again presume for a moment (as was done earlier in defining the dual objective function) that the sectors are made to pay a penalty for the surplus allocated to them as the result of over-estimation of resources and quotas. Under such circumstances, biased evaluation, i.e., a biased shadow price system, would result in a grave loss to the sector. The sectors would then obviously strive to make this loss as small as possible.

It may thus be seen that both sides strive to reduce a specific kind of relative loss. The centre's aim is that as little as possible should be lost of the optimal capabilities of the economy, the sector's aim is that the optimal evaluations should be surpassed by as little as possible. The minimax solution is achieved when both players succeed in eliminating this relative loss.

(3) Corresponding to Theorem 3 of this paper, a certain leveling trend of the shadow prices appears in the concrete model. First, the demand shadow price (ζ_{jit}) of the same product i is equalized between the various sectors, as are the demand and supply shadow prices (ζ_{jit} and ρ_{it}) of the product. Second, the shadow price (ω_{it}) of the manpower quota allocated to the various sectors is equalized.

This trend fully complies with the familiar optimum condition of "welfare-economics," according to which the utilization of the marginal returns of identical resources must be equal in the various spheres.[31]

The equalizing trend is, of course, only valid for shadow prices related to the same period. (It will be worth studying the ratios of the shadow prices of consecutive periods, for they will make it possible to determine a group of "discount rates.")

(4) As has been pointed out, the economic policy figures of the present model are taken from the original plan, worked out by "traditional methods." Moreover, this original plan may also be chosen as the initial program for iteration. The first steps of the iteration should reveal whether or not the plan is realistic. If artificial variables (fictitious unbounded imports) appear in the sector programs, the original plan was not balanced, but the further steps of two-level planning will serve to balance it. If, however, it is not possible in the course of subsequent steps in two-level planning to eliminate the fictitious variables, then this is a warning that there is a contradiction in the economic policy figures.

Two-level planning thus offers an opportunity to carry out a critical check of the original plan, to discover and obviate any contradictions it may contain. As the equalization of the shadow prices of the central directives is approached (e.g., as an approximately accurate knowledge of the OCI shadow prices of the external consumption Q_{it} is obtained), so will further information become available for the critical evaluation of the economic policy figures adopted from the original plan (e.g., to decide whether it would not be opportune to set out from a different pattern of external consumption).

(5) One of the most problematic features of this concrete model is the economic content of the objective function. The optimization of the foreign trade balance as

[31] See, for instance, the works of Samuelson [22] and Lerner [16].

an optimum criterion is in the present description intended as nothing more than an example. In the course of discussions of this problem in Hungary, other ideas have also been advanced, e.g., the minimization of total manpower expenditure, or the maximization of external consumption according to a given pattern. (This is the type of objective function recommended by Kantorowich in his work [9].) In both cases the economic policy targets relating to the trade balance must be incorporated in the constraint system.

It is not intended in this paper to take a definite stand on this problem; it requires a many-sided theoretical and practical investigation. At any rate, in the case of the first experimental computing projects, it will be advisable to use several kinds of objective functions and to compare the results.

(6) Finally, there is another grave problem which can be no more than mentioned: the expression of society's time preference in the model. This is partly circumvented by prescribing external consumption separately for each period (naturally seeing that it should increase for each consecutive period, and that its pattern should change in the required manner). It is not, however, a matter of indifference as to when the surplus returns obtained as a result of the programming will arise—whether this is to be earlier or later. It may therefore be advantageous not simply to maximize the sum of all the returns for the whole plan period, but rather to maximize some discounted total.

The other difficult question is linked to the finite duration of the plan-term. The structure of the model as described above may involve the danger of having the program prescribe only investments whose returns appear within the plan-term. This aspect would only be solved by planning for an infinite duration, but this device, because of other considerations, is not yet practicable. For this reason, as an approximation (or we might say, by way of a compromise), the following solution was chosen.

The requirements (Q_{it}) of external consumption are made to include the needs of the so-called "carry-over investments," i.e., those that will continue after the end of the plan-term. For lack of any other source, the estimates of these figures are again adopted from the original plan. The authors are well aware of the problematic features of this solution, and the question will therefore continue to be investigated.

CONCLUSION

By way of conclusion, the following is a brief summary of the further trends of our research:

(1) Our mathematical and computing research is directed mainly at elucidating how convergence in the course of two-level planning can be accelerated. Numerical experiments are being carried out to this end. It will require special study to determine whether, in the case of the concrete model, economic information that is available elsewhere could not be used to accelerate convergence.

(2) Parallel with the numerical experiments, preparations for the practical application of the concrete model have been begun. The National Planning Bureau wishes to make use of this method too, to obtain sounder foundations for long-term macroeconomic plans. It must, of course, be stressed that these calculations are at present no more than experimental. They are in the stage of scientific research and can only gradually become permanently used instruments of planning.

For a more exact representation of the complicated economic interconnections, the structure of the model applied in practice, is in some respects different from the simple model described in Section 2 of this paper. (E.g., a sector has not one, but several products; there are several primary factors, etc.)

(3) The general model described in Section 1 of the paper may also be concretely applied to other practical problems. Thus, for instance, the determination of the short-term plan of Hungarian cotton fabric exports (their composition by products and markets) is now under preparation, using a method analogous to the two-level planning one.[32] It is also intended to use the method for the elaboration of regional plans—in this case each sector corresponds to a geographic region. The authors hope that once the computing problems have been solved, the method can be widely applied.

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